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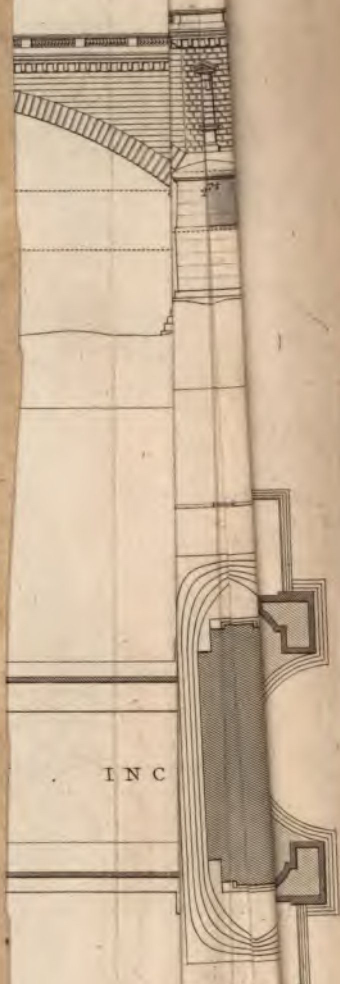

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No 260

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W Perrell

TREATISE

ON THE

EQUILIBRIUM OF ARCHES,

IN WHICH

THE THEORY IS DEMONSTRATED

UPON

FAMILIAR MATHEMATICAL PRINCIPLES.

SECOND EDITION.

By JOSEPH GWILT, Architect, F.S.A.

LONDON:
PRIESTLEY AND WEALE.
MDCCCXXVI.

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TREATISE

ON THE

EQUILIBRIUM OF ARCHES.

them for selection by the Court of Common Council of the city of London. That Court, nevertheless, had not sufficient liberality to reward the Design in question. To a young artist their conduct might have been ruinous, and it was fortunate for the Author, that his dependence was not on such patrons. The sapient body in question have however been, since that time, sufficiently chastised for their mismanagement, by a committee of the House of Commons, which wisely took from them all control over the rebuilding of the bridge, and placed it in more independent hands.

J. G.

Abingdon Street, Westminster,
January 1, 1826.

INTRODUCTION.

An arch is an artful arrangement of bricks, stones, or other materials, in a curvilinear form, which, by their mutual pressure and support, perform the office of a lintel, and carry or sustain superincumbent weights, the whole resting at its extremities upon piers or abutments.

The construction of an arch is one of the most important and curious operations in the science of architecture; it enables the artist to carry roads and canals over the widest valley, or most rapid river, to resist the greatest pressure, and at the same time to impart a light and beautiful character to his work.

An investigation of the pressures that take place amongst the materials whereof an arch is composed, and the method of arranging them so that the whole may remain at rest, are the subjects of the following work.

It does not appear by the remains of buildings still existing in Italy and other places, that the ancient architects, in the construction of arches, were guided by equilibrial principles. They seem to have been unacquainted with any rules for resisting the displacement which the voussoirs are subject to by their mutual pressures, or for counteracting the lateral thrust of the whole arch against the piers. Experience and a certain mechanical perception seem alone to have guided them. We often find the piers of their bridges equal in thickness to nearly half the span of the arch. In other similar cases not a fourth part of the span is assigned to them. Had they proceeded upon fixed and decided rules or principles, this difference would not have existed.

Vitruvius is very particular with regard to the methods employed in building walls, bonding them, &c. and would, it may be presumed, have mentioned the principles upon which they proceeded in building their arches, had any then existed.

Probably, says Bossut, this part of the science of building, as well as the other constructive branches of it, was abandoned to the artificers, and the architect troubled himself about little besides the design and general distribution.

To the architects of the middle ages we are indebted for the noblest specimen of arched roofing. The skill employed in counteracting the thrusts, and balancing the pressure of the different arches in the cathedrals, and other remaining structures of our forefathers, exhibits a proof that the fraternity of freemasons had an accurate knowledge of this complex and difficult branch of the art.

It is not the object of this introduction to trace the history of the arch from its origin through its various modifications. The progress must necessarily have been slow, and marked by many degrees of improvement. Two stones inclined to, and meeting each other, required some care so to adjust that they should not push out the props or walls upon which they

rested. If, instead of letting the stones meet each other, a third stone be interposed, after the manner of a key-stone, we have the element of an arch. The conditions now change, and more adjustment will be necessary. Add a fourth stone, and still more science is requisite to prevent their fall.

It was comparatively late that the theory of arches attracted the notice of mathematicians. Dr. Hooke gave the hint, that the figure of a perfectly flexible cord or chain, suspended from two points, was the proper form for an arch.

Galileo considered the catenary as a parabolic curve, and John Bernouilli appears to have been the first who discovered its nature. Dr. Gregory (Phil. Trans. 1697) published an investigation of its properties, and observes that the inverted catenary is the best form for an arch on account of its lightness*. This is true so

* "Catena in plano verticali, sed situ inverso, figuram servat nec decidit, adeoque arcum seu fornicem facit tenuissimum."

long as it is not pressed by an extraneous weight. It is not, however, capable of bearing a load on any part, much less of being filled up on the spandrels, which must be the case in practice. Other considerations must be involved before it can be fitted to receive a roadway or other weight, either upon its crown or haunches.

In 1695, La Hire, in his "*Traité de Mecanique*," established by the Theory of the Wedge, a method of determining the proportion in which the lengths of the voussoirs should increase as they approximate the springing.

The historian of the Academy at Paris intimates, in 1704, that Parent following the same theory, but without proceeding further than mechanical delineation, gave the extrados of a semi-circular arch, together with its horizontal drift.

Couplet wrote two memoirs in the volumes of the French Academy for the years 1729 and 1730. He trod in the steps of La Hire and Parent, and made but little advances in the

scientific investigation of the subject. He treats chiefly on the drift or shoot of semi-circular arches and the thickness of the piers, considering the arch-stones as infinitely smooth, and experiencing no resistance from friction.

In the volume of the Transactions of the French Academy for 1734, is a memoir by Bouguer, "*Sur les Lignes Courbes propres à former les Voutes en Dôme.*" In this is an analogy between cylindrical and dome vaulting, the one being supposed to be formed by the movement of a catenarean curve parallel to itself, and the other by the revolution of the same curve about its axis.

The wedge theory has since been followed by Belidor and others on the Continent, and the late ingenious Mr. Attwood in this country. It must not, however, be confounded with one of which a notice may be found in the Transactions of the Royal Irish Academy for the year 1789, in a memoir by Young on the Gothic arch, nor with the hint of one given by Dr. Gregory in

his Mechanics, second edition, 1807, the results of which, if carried through, would correspond with those in the following treatise.

Such was the state of the progress towards a perfect developement of the true theory of arches, when our countryman Mr. Emerson in 1743 investigated the nature of a proper extrados of the different curves. Casting a new light upon this curious and useful subject, he exposed the fallacy of La Hire's wedge theory, and it is now completely exploded.

Dr. Hutton, in his "Principles of Bridges," and in the first volume of his tracts, pursued the subject on the principles laid down by Emerson. He added the method of finding an intrados to a given extrados, which indeed seems the principal addition he made to this branch of science.

Bossut, on the Continent, latterly republished with additions, in his "*Cours de Mathematique,*" a tract on the subject; and in 1785, Mascheroni

published at Bergamo, a work intituled "*Nuove Ricerche sull' Equilibrio delle Volte,*" in which there are some propositions upon the equilibrium of arches, but it principally relates to domes on circular, elliptical, and polygonal plans.

It now remains for the Author of the following treatise to state the reasons which first induced him to bring it before the public.

The works of Emerson and Hutton are elegant, as might be expected from the pens of such excellent mathematicians, but they are nevertheless of little if of any service to a profession, whose province it is to practise that science, whereof a just knowledge of the theory of arches forms but a small proportion. The fluxionary calculus is used by both authors; this is little understood by the profession in general, nor indeed is it absolutely necessary that it should be. The analytical process to any other person than a mathematician, is not so striking, as a

demonstration by means of lines, where the operations constantly strike the sense*.

It therefore appeared to the Author, that a short Treatise, founded on the theory of Emerson, wherein the demonstrations did not require an acquaintance with the higher branches of the mathematics, would be profitable to the students of architecture, and perhaps even to the professors. It is not, however, meant to be inferred that the architects of this country are destitute of mathematical acquirements; but it must be recollected, that many sciences are connected

* Rousseau, speaking of the application of algebra to geometry (Confessions, Liv. 6,) says, "Je n'ai jamais été assez loin pour sentir l'application de l'algebre à la geometrie. Je n'aimois point cette manière d'operer sans voir ce qu'on fait; et il me sembloit, que resoudre un problême de geometrie par les equations c'etoit jouer un air en tournant une manivelle.—Ce n'etoit pas que je n'eusse un grand gout pour l'algebre en n'y considerant que la quantité abstraite; mais appliqué à l'etendue, je voulois voir l'operation sur les lignes, autrement je n'y comprenois plus rien."

with their profession, and that a perfect knowledge of each cannot be expected of them *. Neither is it intended to disparage the analytical art, the importance and utility of which, in investigations similar to those above alluded to, is obvious.

Under the foregoing circumstances, the Author flattered himself that the following pages would meet with that indulgence from mathematicians to which, from the nature of his avocations in business, he conceived himself fairly entitled. Such has been the case; and in this second edition, in which many important alterations and

* Vitruvius, after enumerating the many qualifications of an architect, says, "that he need not be singularly excellent in the sciences, though he should not be entirely without a knowledge of them; for in such a number of different things, it is not possible to attain to singular perfection in each."— "Nor is it architects alone who cannot arrive at this pitch of eminence in all literature; for even some of those who have professed a single art, have not been able to obtain the highest degree of reputation therein." Book I. Chap. I.

additions have been made, he again trusts that his endeavours to elucidate and render more familiar an interesting and highly important branch of science, will be received by the profession of which he is a member, as a token of respect and esteem.

A

TREATISE

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SECTION I.

Of the general Laws of Motion and the Composition and Resolution of Forces.

DEFINITION I.

WHATEVER changes or tends to change the condition of a body, with respect to rest or motion, is called force: thus, pressure, impact, gravity, &c. are called forces.

DEFINITION II.

Momentum signifies the quantity of motion in a body, and is measured by the velocity and

quantity of matter jointly: thus, if the quantity of matter in A is represented by 10, and that in B by 12, and their velocities by 15 and 11 respectively, then the ratio of their momenta is as 10 multiplied by 15 is to 12 multiplied by 11 or as 150 to 132.

DEFINITION III.

Mechanical action is that of a moving force, and its effect is measured by the momentum generated in a given time.

AXIOMS.

1. Every body will continue in its existing state, either of rest or uniform rectilinear motion, until acted upon by some force tending to change its state.

This law is conformable to our ideas of matter and motion, for where no change of condition in a body is supposed, no power is supposed to act; and when a change is supposed, then also some power is supposed to act. The law is usually illustrated by this familiar circumstance, that when a man is standing in any vehicle, if the vehicle suddenly change its state from rest

to motion, or the contrary, the man is in danger of being thrown from his actual position.

2. Every motion or change of motion in a body, is proportional to the force exerted to produce it, and is in the direction of the right line in which such force acts.

Though this law also depends upon our ideas of matter and motion, yet its truth is also inferred from experiments on the collision of bodies; and though some cases occur in which the velocities generated by forces are not, simply, proportional to those forces, it is supposed that, in such cases, only parts of the forces are effective in producing the observed velocities.

3. Action and re-action between any two bodies are always equal and opposite in direction.

By action and re-action equal changes of condition are produced in bodies acting on each other: for, let A be a simply hard body, as a ball moving with a given velocity, and let it impinge upon another equal ball at rest; the two balls will, after impact, move together with half the velocity which A had before impact. It is evident therefore that A communicates to B a velocity equal to half its own velocity, and

is, by the re-action of B, deprived of just so much velocity as it has imparted to B. Thus the momentum which is measured as above, remains unaltered.

Having defined what is meant by a force, and shewn the most general laws of its action, we proceed to state the circumstances attending the joint action of two or more forces upon a body subject to their influence. Two forces may act upon a body in three different ways; they may act in the same, or in opposite directions, or they may act obliquely to each other.

If two magnitudes, as two lines or two numbers, be taken to represent two forces, the sum of those magnitudes must represent a force which is equivalent to the two given forces when they act in the same direction; and the difference of the magnitudes must represent a force equivalent to the forces when they act in opposite directions: so that in this case, when the forces are equal, the difference of the magnitudes being nothing, shews that an equilibrium subsists, or that the forces destroy each other's effects.

But if the forces act obliquely on a body, it is evident that there must exist one single force capable of producing the same effect upon the

body as the given forces; and this is what we purpose next to determine.

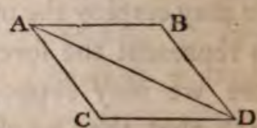
This proposition and its converse, which form the subject of the composition and resolution of forces, admit of demonstration, from the consideration of the forces themselves, without requiring the idea of motion to enter into the investigation; but the great intricacy of such demonstration* renders it preferable to avail ourselves of the second law of motion; viz. that the motion produced in any body is proportional to the acting force: in which case we may employ the motions, or spaces described, to represent the forces themselves; and therefore we may suppose the given forces to be such that they would cause the body to move with similar motions through certain spaces respectively in the same time, and we shall then only have to determine the space described in the same time by the given forces acting together. This being understood, we proceed to state the following propositions.

* See Robinson's Mech. Philos. or Gregory's Mechanics.

PROPOSITION I.

If a body at A *be urged by two similar forces, so that they would separately cause it to move along the adjacent sides* A B, A C, *in an equal time: then will those forces, communicated at the same instant, cause it to pass through the diagonal* A D, *of the parallelogram* A B C D, *in the same time.*

For the force (Axiom 2.) along the side A B can make no alteration in the body's approach to C D, to which A B is parallel, since it will arrive at C D



in the same time that it would have done had no motion been communicated in the direction A B. In the same manner it will arrive in B D in the same time that it would have done had no motion been communicated to it in the direction A C. It will therefore be found at the intersection D of the sides C D and B D. And as the forces are supposed similar, the body will move with a rectilinear motion, and will consequently pass along the diagonal A D of the parallelogram A B C D.

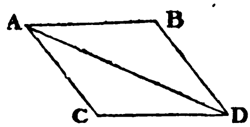
COROLLARY.

If two sides of a triangle represent the spaces over which two forces would separately carry a body in an equal time, the third side will represent the space over which these forces acting conjointly would urge it uniformly in that time. For if A C and C D be the two sides; by completing the parallelogram A D, the third side will be the diagonal thereof.

PROPOSITION II.

If A B, A C, *the adjacent sides of the parallelogram* A B C D, *represent the quantities of two forces acting upon a body at* A, *as also the directions of those forces, then will the diagonal* A D *represent a force equivalent to them.*

By the preceding proposition, A D is the space described when the body is urged from A by two similar forces at the same instant; and as A B, A C, and A D, represent the spaces passed over



in equal times, they also represent the momenta; and as the spaces or the momenta may be taken

to represent the forces themselves, for the reason before given, consequently the forces A B, A C, compounded or acting on A at the same instant, exert a force whose equivalent is A D.

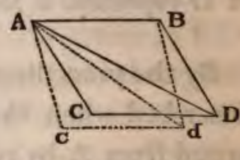
COROLLARY I.

If two sides of a triangle represent in quantity and direction two forces acting on any body, then will the third side represent a force equivalent to them both, acting conjointly.

COROLLARY II.

As the angle in which two forces act is diminished, the compound force is increased.

Let A B, A C, represent two forces acting in the lesser angle C A B. and A B, A c, two other forces respectively equal to the former, and acting in the greater angle c A B.



It is manifest that the angle A B d is less than the angle ABD, and the sides B D, B d, being equal, and A B common to both; therefore (Euc. Prop. 24, B. 1.) the base A D, of the triangle A B D, will be greater than the base

A d, and A D is the equivalent of the two forces acting in the lesser angle.

COROLLARY III.

Two forces acting in the same direction, produce the greatest effect.

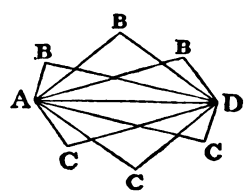
COROLLARY IV.

A body cannot be kept at rest by two forces, unless they are equal and in opposite directions.

PROPOSITION III.

A single direct force A D, *may be resolved into any number of oblique forces.*

For the direct force A D may be resolved into as many pairs of forces as there may be triangles described upon it, as A B D, &c. &c. or parallelograms A B C D, &c. &c.; to all the pairs of which separately, A D will, by the last proposition, be equivalent.



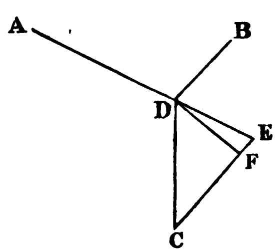
COROLLARY.

If two forces are together equivalent to A D, and B D be one of them, A B must be the other.

PROPOSITION IV.

If three forces A, B, C, *acting together be in equilibrio, they are proportional to the three sides* D E, E C, C D, *of a triangle drawn parallel or perpendicular to the directions of those forces.*

Let C D, A D, and B D, represent the direction of the forces. Produce A D indefinitely, and at C parallel to D B, draw C E meeting A D produced in E. Then D C, E C, and E D, represent the three forces.



For take D C to represent the force in that direction; then if E C do not represent the force in the direction D B, let F C represent it, and join F D. Then (Prop. 2. Cor. 1.) if D C,

C F, represent the quantities and directions of any two forces, F D will represent the quantity and direction of their equivalent. But as D C, C F, represent the forces in C D, B D, and as the two latter are kept in equilibrio by the force in A D, and that by hypothesis, D F is the equivalent of those latter; it follows that D F is equal and opposite to A D, or A D, D F are in the same straight line. Hence the straight lines A D E, A D F, have a common segment, which is impossible (Euc. Cor. Prop. 11. B. 1.) Therefore E C represents the force in the direction D B, and consequently E D represents the third force; and any three sides of a triangle drawn parallel to the directions of three forces A D, D B, D C, acting together, are proportional to the sides of the triangle D E C, and consequently proportional to the three forces.

Again, since if three lines be drawn perpendicular to the three sides of a triangle, they will form a triangle similar to that given; it follows, that, if three forces are in equilibrio about a point, they will be proportional to the sides of a triangle drawn perpendicular to the directions of those forces.

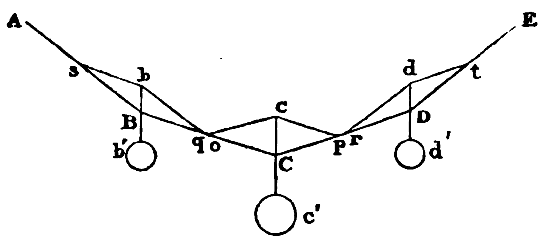
COROLLARY I.

Because the three sides of a triangle are proportional to the sines of their opposite angles, therefore three forces, when in equilibrio, are proportional to the sines of the angles made by their lines of direction.

COROLLARY II.

If one of the forces, as C, be a weight sustained by the two strings D A, D B, the force or tension of the string A D, is to C, or the tension of the string D C, as D E to D C, and the tension of B D is to C, as C E to C D.

LEMMA.



Let A B C D E be a thread stretched throughout in the direction of the different parts of its

length A B, B C, C D, D E, (the ends A, E, being fixed) by means of weights suspended from the angles B, C, D, acting vertically; and let the weight c′ be represented by the vertical C c. Consider such vertical as the diagonal of a parallelogram, whose sides are parallel to B C and C D, and complete the same. Make B q and r D respectively equal to o C and C p, and from the points q and r, on the indefinite verticals b B, d D, complete parallelograms, whose sides are parallel to A B, B C, and C D, D E. Then B b, D d, will represent the weights which must be suspended at the points B and D to preserve the figure, and B s, B q, D r, D t, will represent the contractile forces or tensions at B and D in the directions of those lines respectively. The truth of this lemma is evident from Prop. 2. For B b, C c, D d are forces equivalent to those represented by B s, B q, C o, C p, &c. respectively, and because B q = o C, C p = r D, &c. the tensions at B and C, C and D are equal in opposite directions. Therefore the figure subsists in equilibrio.

If the figure be inverted, and considered as consisting of props instead of strings, the same equilibrium will obtain, the weights being

brought to act in a vertical direction on the angles.

Props inverted act as strings, and the converse; the former suffering compression, and the latter tension.

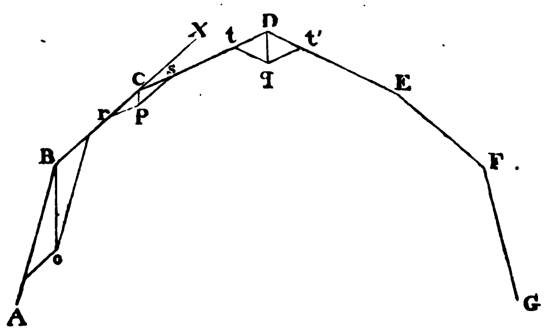
If A B C D E were a thread or chain attached to immoveable points at A and E, and left, without any accidental weights, to the action of gravity, it would assume the form of that particular curve which has been called a simple catenary; and if a system of indefinitely small and equal bodies were arranged in the same form, but in an inverted position, it would also stand in equilibrio, and such a figure would exhibit what might be called an arch of equilibration; such arch, however, would be of no practical use, because it could not be loaded. The case would be different if the catenary were of the kind called complex, as we shall presently see.

PROPOSITION V.

PROBLEM.

The lines A B, B C, C D, *are inclined to each other in angles given or known, and the weight*

placed over the angle D *is also given; required the weights which must be placed on the other angles* B *and* C, *so that the whole assemblage may remain in equilibrio,* A *and* G *being fixed.*



Describe parallelograms, and proceed as in the foregoing lemma, D q being the weight, or that which represents the weight at the angle D.

Then (Prop. 4. Corol. 1.) those forces which balance the lines at the angles B, C, D, &c. are as the sines of the angles of a triangle, formed by their lines of direction, and in the parallelogram C r p s, C p expresses the force at C, acting vertically, and C r, C s, the equivalent forces into which it is resolved.

1. Produce r C to X, then will ∠ s C X =

$\angle$ p r C, and as sine $\angle$ r C s = sine $\angle$ s C X, it is also equal to sine $\angle$ p r C, to which C p is opposite.

2. r p is opposite to $\angle$ r C p, and is equal to C s or t D.

3. The angle r p C = $\angle$ p C s and r C is opposite to $\angle$ r p C, whose sine is consequently equal to the sine of $\angle$ p C s.

Therefore the three forces C p, C s, and C r, are respectively proportional to the sines of the angles p r C, r C p, and p C s. From whence may be deduced, that the weight on any angle is directly as the sine of that angle, and reciprocally as the sine of the two parts into which it is divided by a vertical line, as follows. The force C r : force C s :: sine $\angle$ p C s : sine $\angle$ r C p. But s $\angle$ p C s = s $\angle$ q D t, since those angles are supplements of each other. Then by the nature of proportion, the force C r : force C s :: $\frac{1}{s \angle r C p} : \frac{1}{s \angle q D t}$. In the same manner it may be proved, that the force t D : force D t$'$ :: $\frac{1}{s. \angle q D t} : \frac{1}{s. \angle q D t'}$ and so on. Whence it is evident that the forces C r and D t vary as $\frac{1}{s. \angle r C p}$ and $\frac{1}{s. \angle q D t}$.

Now, in the triangle C p r, from the proportion of the sides to the sines of the opposite angles, we have $C\,p = \frac{s \angle C r p \times C r}{s. \angle C p r}$. Therefore the weight represented by C p will be proportional to this expression; and because C r varies as $\frac{1}{s. \angle r C p}$, if we substitute this last term in the above value of C p, we shall have the weight or force C p proportional to $\frac{s. \angle C r p}{s. \angle C p r \times s. \angle r C p}$; but s. ∠ C r p = s. ∠ X C s, or s. ∠ r C s. Therefore C p is proportional to $\frac{s. \angle r C s}{s. \angle p C s \times s. \angle r C p}$.

We shall therefore have the following proportions for finding the weights or lines representing them; viz.

$$\frac{s. \angle tDt'}{s. \angle tDq \times s. \angle qDt'} : Dq :: \frac{s. \angle rCs}{s. \angle rCp \times s. \angle pCs} : Cp.$$

In numbers thus:

Let ∠ C D E = 130° · 00′
∠ B C D = 165° · 00′
∠ A B C = 145° · 00′

And let the weight or force represented by D q = 1·00; then, because in the figure we have supposed ∠ C D E to be bisected by D q, we

shall have $\angle$ q D t′ = 65°; and because $\angle$ p C s is the supplement of $\angle$ q D t, or $\angle$ q D t′, we have

$$\angle\ pCs = 115°.\ 00'$$
$$\angle\ rCp = (165° - 115°) = 50°.00'$$

We may now determine the value of C p, B o, &c. in which the operation would be facilitated by using a table of logarithmic sines: but, to be more intelligible, we shall take the values of the trigonometrical terms from a table of natural sines, the labour of calculating such sines being too troublesome for practice. Thus,

The natural sine of $\angle$ C D E (= 130°) or of $\angle$ r C p (= 50°), these angles being supplements of each other } = 7660444

The nat. sin. $\angle$ q D t′ or $\angle$ t D q (= 65°) or of its supplement $\angle$ p C s (= 115°) } = 9063078

The nat. sin. $\angle$ BCD (= 165°) = 2588190

Whence

$$\frac{7660444}{9063078 \times 9063078} : \frac{2588190}{7660444 \times 9063078} ::$$

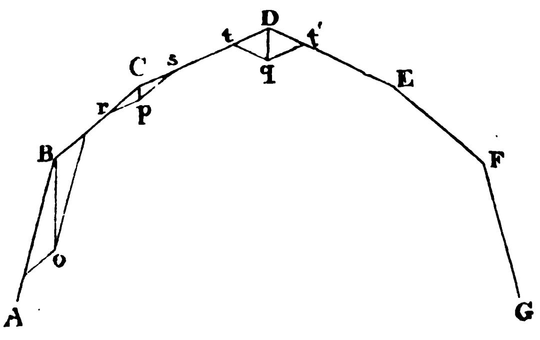
1·00 : ·39972 nearly the value of C p.

In like manner B o is found to be 3·1019 nearly.

PROPOSITION VI.

PROBLEM.

The weights or lines representing them, D q, C p, B o, *being given, as also the direction of the first line* D C, *or the angle* C d q, *which it makes with the vertical* D q; *it is required to find the direction of the other sides, so that an equilibrium between the parts of the assemblage may take place.*



Complete the parallelogram upon the diagonal D q, make C s equal to D t, join s p, and make B C parallel thereto; then we shall have

the angle B C D, or the direction of the next side B C. Proceed in the same manner for the angle at B, and the direction of the side A B will be obtained.

Let the weights $D\,q = 1{\cdot}00000$
$C\,p = 0{\cdot}39972$
$B\,o = 3{\cdot}10190$

Let the angle q D t, which the first line makes with the vertical, $= 65^\circ{\cdot}\ 00'$.

In the triangle D t q, by the proportion of the sides to the sines of the opposite angles, we have s. $\angle$ C D E, or its supplement D t q : D q :: s $\angle$ q D t : t q, or its equal D t. Therefore

$$D\,t = \frac{D\,q \times s.\angle\,q\,D\,t}{s.\angle\,C\,D\,E}$$

That is, $\dfrac{1 \times s.\ \angle\ 65^\circ{\cdot}\ 00'}{s.\ \angle\ 130^\circ{\cdot}\ 00'} = D\,t = C\,s = 1{\cdot}1831.$

Now the $\angle$ p C s or p C D = the supplement of $\angle$ q D C = $115^\circ{\cdot}\ 00'$. The angle B C D is evidently composed of the angles B C p and p C D, and the angle B C p (Euc. Prop. 29, B. 1) = $\angle$ C p s; we shall therefore have $\angle$ p C s, and the sides C s and C p to find $\angle$ C p s, to which $\angle$ B C p is equal.

And the angles B C p, p C D, are together equal to the angle B C D.

By plane trigonometry,

As, C p + C s : C p − C s :: tangent of half the sum of the angles C p s, C s p : tangent of half the difference of those angles.

Or in numbers,

As, $1{\cdot}1831 + {\cdot}39972 : 1{\cdot}1831 - {\cdot}39972 ::$ tangent $32^{\circ}\cdot 30'$: tangent $17^{\circ}\cdot 30'$.

Therefore $32^{\circ}\cdot 30' + 17^{\circ}\cdot 30' = \angle\, Cps = 50^{\circ}\cdot 00'$.

And $50^{\circ}\cdot 00' + 115^{\circ}\cdot 00' = \angle\, BCD = 165^{\circ}\cdot 00'$.

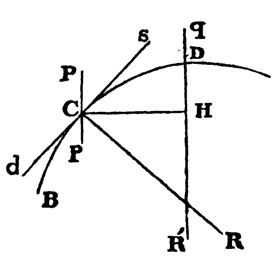
By a similar process we shall find the angle $ABC = 145^{\circ}\cdot 00'$.

If the other side of the figure E F G, be completed, and the fixed points be A and G, the opposite sides will mutually balance, and the equilibrium will be still preserved.

OBSERVATIONS.

The preceding propositions contain the principles of all that is necessary to be known by the practical builder for balancing the arch, be it of whatever form; it will be seen in the following pages that they are extremely fruitful in application, and equally useful in practice.

From the arrangement of lines forming the sides of a polygon, an equilibrium (as we have shewn) has been obtained. These lines may be diminished in length, and their number increased without end, in which case the polygon is exhausted or coalesces with a curvilinear arch, and it is evident that the result will be the same. It may not perhaps be unnecessary to observe, that in algebraical expressions of the equations of curves, and in fluxionary calculations relative thereto, the increments of the abscissæ of the curve are given, and ordinates determined with reference to the deflections of the sides of a polygon, which finally falls in with the curve. The sides of the polygon having been thus supposed indefinitely short, and the polygon exhausted, we shall have a curve B C D, and the angle X C D (see fig. to prop. 5. sect. 1.) whose sine is equal to that of the angle B C D, will become the angle of contact s C D in the figure, or that formed



by a tangent to the curve at any point C, and the curve itself. Now the angle of contact de-

pends upon the degree of curvature of the arch at the point C, and diminishes as the radius of curvature increases, because then the curve approaches nearer to a coincidence with the tangent. Therefore we say the angle of contact is reciprocally proportional to the radius of curvature C R, directions for finding which, in all common curves, will follow in the next section. And the angles p C B, p C D, become equal to the angles p C d, p C s, which are supplements to each other; consequently their sines are equal, and equal to that of the angle p C s. Hence the arch being considered as a system of indefinitely short beams, the weight on an indefinitely small part of it at C, will be reciprocally as the radius of curvature at C, and the square of the sine of the angle made by the tangent to the curve and the vertical; that is as, $\frac{1}{C\,R \times (s.\ \angle\, p\,C\,s)^2}$. Now this would be sufficient for determining the conditions of equilibrium, if it were not that in an arch constructed to carry a load or part of an edifice, the weight must be diffused over every part of the arch, whereas this term expresses only the weight to be applied at the points of junction of the sides of the polygon,

which are considered as beams inflexible and without weight.

Suppose next, the curve of the arch to be divided into any number of equal parts, representing the breadths of the voussoirs or arch-stones in that direction, and to avoid complicating the subject by introducing pressures oblique to the curve, let the joints of the voussoirs all tend to the centre of curvature of the arch, or be perpendicular to the tangents at the places where the joints are situated. Then, from the nature of all curves, the horizontal breadths of the voussoirs increase from the springing part up to the vertex or crown. Now, every point on the surface of the voussoirs ought to be pressed by a weight proportional to the value of C p just mentioned, but the vertical direction of the loading causes that loading to press on fewer points of the inclined voussoirs than of the horizontal one at the crown, and the points of pressure will be fewer as we approach the springing course. Therefore this deficiency of pressure on the inclined voussoirs must be compensated by a greater height of loading, and it is evident that the height, on this account, must be inversely proportional to

the breadth of the column over each voussoir; that is, to the horizontal distance between two verticals. But it is evident that these breadths are directly proportional to the sines of the angles between the tangents and verticals; that is to s. $\angle$ p C s, for a column at C. It follows therefore that the heights of the verticals must, on this account also, be inversely proportional to s. $\angle$ p C s: consequently the height C p, must, on all these accounts together, be proportional to $\frac{1}{CR \times (s. \angle pCs)^3}$.

Or since $\angle$ p C s is equal to the complement of $\angle$ s C H which is the inclination of the curve at that point, to the horizon, we have $\frac{1}{s. \angle pCs} = \frac{1}{cos. \angle sCH}$, which by trigonometry, is equal to sec. $\angle$ s C H.

Therefore we have C p reciprocally proportional to the radius of curvature, and directly proportional to the cube of the secant of the curve's inclination, at that point to the horizon. But at D the inclination being nothing, its secant is equal to radius or unity: consequently the proportion for finding C p becomes

$\frac{1}{D\,R'} : q\,D :: \frac{(\sec. \angle\, s\,C\,H)^3}{R\,C} : C\,p$; where D R′ and R C are radii of curvature to the points D and C.

The above is a general expression applicable to all the curves that are capable of being applied in practice.

In order to keep in view the principle of the catenary; let it be observed that a thread or chain may be supposed attached at both ends to fixed points, and that from every part of the curve it assumes, chains may be attached, having their lengths so adjusted as, by their weight, to drag the original chain into any form whatever, suppose a semi-circle, semi-ellipse or any other curve, forming what may be called a complex catenary. Then if an arch of the same form were erected in a vertical plane, and had a loading above it equal in height, over every part of the curve, to the lengths of the chains attached to the principal one; such an arch with its loading would stand and be in a state of equilibration.

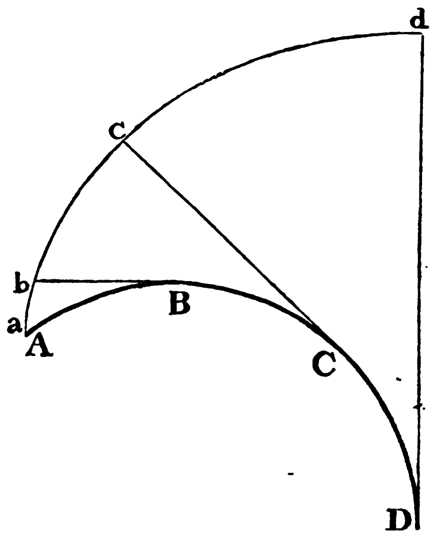
Now the formula just investigated shows how to determine the height of such loading over every part of an arch in a general way. Its application to particular curves will follow in the third section.

As the radii of curvature enter into the general expression of the height of the loading, it will be necessary to shew how these may be determined for the given curves before we proceed further.

SECTION II.

DEFINITION.

Of the Osculatory Circle, and the Method of finding the Radius of Curvature.



Let A B C D be a mould of wood, or other hard material, and let a thread be fixed to it at D, and kept close to its convexity, along the whole extent A B C D. Now taking the

thread by its end at A, and keeping it tort, let it be gradually unlapped or evolved from the mould. Then will its extremity A, describe another curve a b c d, called the involute curve of A B C D, or its evolutrix; A B C D being the evolute curve.

It will be plainly perceived, that the involute curve has a momentary centre at every point in A B C, from which it is unwound; for instance, when it has been unlapped as far as B, and A has arrived in the same time at b, then B is a centre to b; and if with the radius b B, there be described a circle touching the curve a b in b, it will be the osculatory circle, or circle of curvature in that point, and b B will be the radius of curvature.

All curves, a circle excepted, whose curvatures are neither infinitely great nor infinitely small, may be thus described by a thread evolved from a proper curve.

It is inferred that the unwrapped part of the thread is always a tangent to the curve A B C, and that it is perpendicular to the tangent of curvature at b.

In a circle the curvature is every where the same, so that the radius of curvature is constant; and the curvatures of different unequal circles

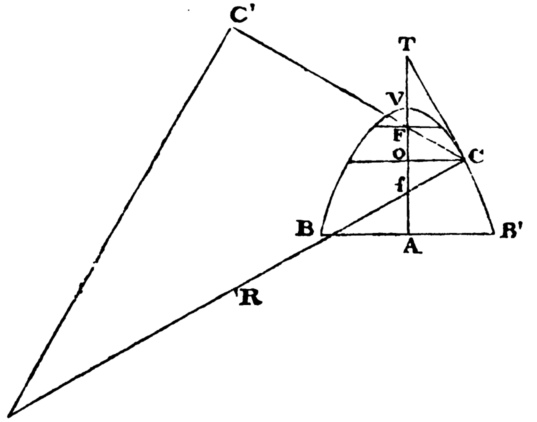
are reciprocally proportional to the diameters; that is, if one diameter be half the length of another, the curvature or convexity of a circle, having the former diameter, will be twice as great as that of the latter.

The demonstrations of the following propositions may be found in Vince's Fluxions and Conic Sections.

PROPOSITION I.

PROBLEM.

In any parabola B V B′. *To find the diameter and radius of curvature to any point* C, *the ordinate* O C *to such point, being given, as also any other ordinate* A B, *and its corresponding abscissa* V A.



Geometrically.

Draw the tangent C T to the point C, and find the focus F*. Through F produce C F in-

* To draw the tangent C T, let O C be drawn perpendicularly to the axis V A, and produce V A above the vertex indefinitely; make V T equal to O V, and join C T, and it will be a tangent to the parabola at the point C.

If C f be drawn perpendicular to C T, cutting the axis

definitely, in which take C C′, equal to four times C′ F; at C′ and C, draw the lines C D, C′ D, at right angles to T C and C′ C respectively, intersecting each other in D. Bisect D C in R; then will D C be the diameter, and R C the radius of curvature to the point C.

Arithmetically.

Let the ordinate A B′ $=7{\cdot}100$.

Its corresponding abscissa V A $=10{\cdot}000$.

Ordinate O C to the point C $=4{\cdot}450$.

Then $A\,B^2 : V\,A :: O\,C^2 : V\,O = 3{\cdot}928$.

And V A : A B′ :: A B′ : P (the parameter) $=5{\cdot}041$.

$$\frac{P}{4} = V\,F = 1{\cdot}26025.$$

$$V\,O + V\,F = F\,C = 5{\cdot}18855.$$

*$\dfrac{4\,(F\,C^{\frac{3}{2}})}{\sqrt{V\,F}} = D\,C = 42{\cdot}1114$, the diameter of curvature at C:

and $\dfrac{42{\cdot}1114}{2} = 21{\cdot}0557$, the radius of curvature at C.

in f, and f T be bisected in F, then F is the focus of the parabola.

* $\dfrac{4\,(F\,C)^{\frac{3}{2}}}{\sqrt{V\,F}}$ signifies four times the square root of the third power or cube of F C, divided by the square root of V F.

At the vertex, the radius of curvature is equal to half the parameter.

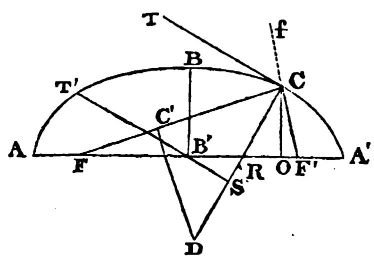
PROPOSITION II.

PROBLEM.

In any ellipsis A B A′, *to find the diameter and radius of curvature to any point* C, *the transverse and conjugate diameters being given, as also an abscissa* A O *to the ordinate* C O *at the point to which the radius of curvature is to be found.*

Geometrically.

Draw the tangent C T*, and parallel thereto, through the centre B′, draw T′ S; join C F, and in it produced, if necessary, make C C′ equal to $\frac{2\,(T' B'^2)^{2}}{A B'}$. At



* From the point B as a centre, with A B′ as a radius, describe an arc cutting A A′ in F and F′, which are the foci; and if lines be drawn from the foci to any point C, and one of them be produced, as F′ C to f, and the angle F C f be bisected, a line drawn from C through the point of bisection will be the direction of a tangent to that point.

right angles to C C′ at C′, and to T C at C, draw C′ D, C D intersecting each other in D; C D is the diameter of curvature required. Let C D be bisected in R, then R C is the radius of curvature.

NOTE.—In this and the following figure, C′ C is taken equal to one half only of the length required, on account of the inconvenient space these figures would otherwise have covered.

Arithmetically.

Let the transverse diameter $A\ A' = 20{\cdot}00$,
the semiconjugate $B\ B'' = 5{\cdot}80$,
Abscissa (to C O) $A\ O = 16{\cdot}00$,
Consequently $B'\ O = 6{\cdot}00$.

Then by the properties of the ellipse:

$A'\ A^2 - (2\ B'\ B)^2 = F'\ F^2$, and $F'\ F = 16{\cdot}2924$, nearly, which will be the distance between the foci:

And $\frac{F'\ F}{2} + B'\ O = F\ O = 14{\cdot}1462$.

I. To find C O.

$$\left.\begin{array}{r} A\ B'^2 : B'\ B^2 :: A\ O \times O\ A' : C\ O^2 = 21{\cdot}5296 \\ \text{therefore } C\ O = \quad 4.64 \end{array}\right\}$$

II. To find C F.

$$\left.\begin{array}{r} F\ O^2 + C\ O^2 = C\ F^2 = 221{\cdot}6445 \\ \text{therefore } C\ F = \quad 14{\cdot}8877 \end{array}\right\}$$

III. To find C F'.

$$A\,A' - F\,C = C\,F' = \quad 5{\cdot}1123$$

IV. To find S C.

$$* \sqrt{\frac{A\,B'^2 \times B'\,B^2}{C\,F' \times F\,C}} = S\,C = \quad 6{\cdot}6492$$

V. To find T' B'.

$$\sqrt{F'\,C \times F\,C} = T'\,B' = \quad 8{\cdot}723$$

VI. To find D C.

$$\frac{2\,(B'T')^2}{C\,S} = D\,C = \quad 22{\cdot}88$$

the diameter of curvature to the point C.

$$\frac{22{\cdot}88}{2} = R\,C = \quad 11{\cdot}44$$

the radius of curvature to the point C.

When the diameter of curvature to the vertex is sought:

$$\text{Then } \frac{2\,(A\,B')^2}{B\,B'} = D\,C = \quad 34{\cdot}482$$

$$\text{And } \frac{34{\cdot}482}{2} = R\,C = \quad 17{\cdot}241.$$

* $\sqrt{\frac{A\,B'^2 \times B'\,B^2}{C\,F' \times F\,C}}$ signifies that the product of the squares of A B' and B' B, is to be divided by the product of C F' and F C, and the square root of the quotient to be taken for S C.

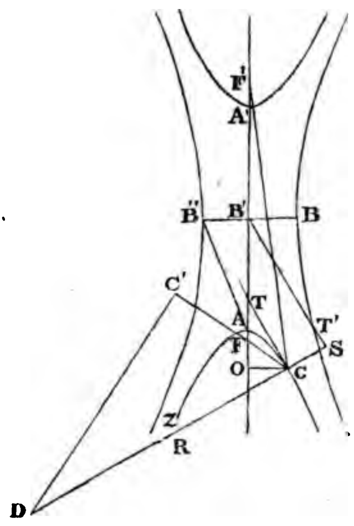
PROPOSITION III.

PROBLEM.

In any hyperbola z A C, *to find the diameter and radius of curvature to any point* C, *the transverse and conjugate axes* A A′ *and* B″ B *being given, as also an abscissa* A O *to the ordinate* O C, *at the point to which the radius of curvature is to be found.*

Geometrically.

Draw the tangent C T*. Parallel thereto, and through the centre B′ draw B′ T′ S; join C F, and produce it indefinitely towards C′. Make C C′ equal to $\frac{2(B'T'^2)}{A\,B'}$, and at right angles to C C′ at C′, and T C at C, draw C′ D, C D intersecting each other in D. C D is the



* Join B″ A and make B′ F, B′ F′ each equal thereto, then F and F′ will be the foci of hyperbola.

diameter of curvature required; and if C D be bisected in R, then R C is the radius of curvature.

Arithmetically.

Let the transverse axis $A\,A' = 15{\cdot}000$,
the semiconjugate $B\,B' = 3{\cdot}104$,
Abscissa (to C O) $A\,O = 5{\cdot}000$,
Consequently $B'\,O = 12{\cdot}500$.

Then by the properties of the hyperbola, $B'\,A^2 + B'\,B^2 = B'\,F^2$. Whence $B'\,F = 8{\cdot}1161$ nearly, which will be the distance of either focus from the centre:

$$B'O - B'F = F\,O = 4{\cdot}3839.$$

I. To find C O.

$$\left.\begin{array}{r} A\,B'^2 : B'\,B^2 :: A\,O \times O\,A' : C\,O^2 = 17{\cdot}1285 \\ \text{therefore } C\,O = 4{\cdot}1386 \end{array}\right\}$$

II. To find C F.

$$\left.\begin{array}{r} F\,O^2 + O\,C^2 = C\,F^2 = 36{\cdot}3477 \\ \text{therefore } C\,F = 6{\cdot}0288 \end{array}\right\}$$

Join F C, F′ C, and bisect the angle F′ C F for the direction of the tangent as in the ellipse.

III. To find C F'.

$$A\,A' + F\,C = C\,F' = 21{\cdot}0288$$

IV. To find C S.

$$\sqrt{\frac{A\,B'^2 \times B'\,B^2}{F'\,C \times F\,C}} = C\,S = \quad 2{\cdot}067$$

V. To find B' T'.

$$\sqrt{F\,C \times F'\,C} = B'\,T' = \quad 11{\cdot}259$$

VI. To find D C.

$$\frac{2\,B'\,T'^2}{C\,S} = D\,C = \quad 122{\cdot}67$$

the diameter of curvature at C.

$$\frac{122{\cdot}67}{2} = R\,C = \quad 61{\cdot}33$$

the radius of curvature at C.

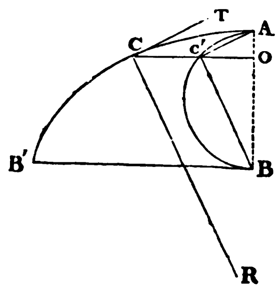
PROPOSITION IV.

PROBLEM.

In any cycloid B' C A, *to find the diameter of curvature to any point* C; *the diameter of its generating circle* A B, *as also an abscissa* A O *to the point* C, *at which the diameter of curvature is to be found, being given.*

Geometrically.

On the diameter A B, describe the generating circle B c′ A, and join C O, cutting the same in c′; join c′ A and c′ B, and parallel to the former at C draw the tangent C T. At right angles to C T at C, draw C R, and make it equal to twice c′ B. Then is R C the radius of curvature at the point C, and twice R C the diameter of curvature.



Arithmetically.

Let A B (diameter of generating circle) $= 5{\cdot}00$

The abscissa A O $= 1{\cdot}00$

Consequently O B $= 4{\cdot}00$

Then, by the nature of the circle

$$\left.\begin{aligned} A\,O \times O\,B = O\,c'^2 &= 4{\cdot}000 \\ \text{therefore} = O\,c' &= 2{\cdot}00 \end{aligned}\right\}$$

$$\text{And } O\,B^2 + O\,c'^2 = B\,c'^2 = 20{\cdot}00$$

$$\text{therefore } B\,c' = 4{\cdot}472$$

$$\text{And } 2\,B\,c' = 8{\cdot}944$$

the radius of curvature to the point C, for C R is equal to twice B c′ by the nature of the cycloid; and

$$4\,B\,c' = 17{\cdot}888$$

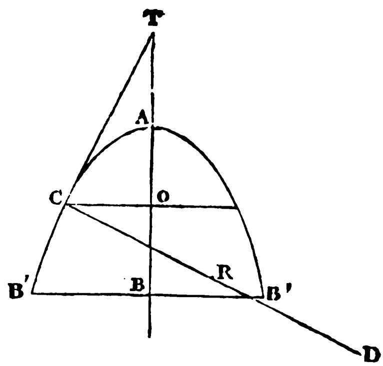
the diameter of curvature.

PROPOSITION V.

PROBLEM.

In any catenarean curve B′ A B′ *to draw a tangent to any point* C, *the span* B′ B′, *and height* A B *of the curve, as also an abscissa* A O *to the ordinate* O C, *at which the tangent is to be drawn being given; and to the same point* C, *to find the radius and diameter of curvature.*

Produce O A towards T, and in it take O T equal to $\frac{CO \times \sqrt{2X \times AO + AO^2}}{X}$, where X is put to represent the tension of the curve at the vertex*. Join T C and it will be the tangent required, and will $= OT^2 + OC^2$.

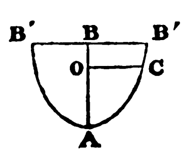


At C draw D C, at right angles to C T, and in it take

* It will be necessary to notice, that in the catenary there

C R, a third proportional to $X : \overline{X + A O}$, that is,

As $X : \overline{X + A O} :: \overline{X + A O} : C R$ (the radius of curvature to C) $= 175{\cdot}37$.



is a peculiar property denominated its tension, that is, the degree of stretching it exerts at any point; and the ordinate to any of its abscissæ cannot be found without involving this tension or stretching in the calculation.

The tension at the vertex A may be found by the summation of the following series, calling it X.

$$X = \tfrac{1}{2} AB \times \left(\frac{B'B^2}{AB^2} + \tfrac{1}{3} - \frac{8 AB^2}{45 B'B^2} + \frac{691 AB^4}{3780 B'B^4} - \frac{33851 AB^6}{453600 B'B^6}\right),$$

&c. which is a series that will serve for every case, taking care to change the real values of A B and B' B, according to the height and semi-span given. Thus,

Let $A B = 40$

$B B' = 50$

Then it will stand thus:

$$X = \frac{40}{2} \times \left(\frac{25}{16} + \tfrac{1}{3} - \frac{128}{1125} + \frac{176896}{2362500} - \frac{97693696}{7087500000}\right), \text{ \&c.}$$

Put these fractions into decimals, to sum them up more easily, and then

$X = 20 \times (1{\cdot}5625 + {\cdot}3333 - {\cdot}1137 + {\cdot}0749 - {\cdot}0138$, &c. $=$ $20 \times 1{\cdot}8432 = 36{\cdot}864$; but if the series had been carried a term further (which is not necessary for practical purposes) it would have come out $36{\cdot}88$, or $36{\cdot}9$ nearly, as Dr. Hutton makes it in his Principles of Bridges (page 37, second edition, London, 1801).

Any ordinate O C to a given abscissa, may be found by a

Hence $2 \times 175{\cdot}37 = 350{\cdot}74$ (the diameter of curvature D C).

In the above the height A B = 50, B′B = 30, and A O = 25.

summation of the following series. Let it be to an abscissa A O = 40·00. The constant tension (= X) being as above 36·9 nearly. Then

$$OC = \sqrt{2 \times X \times AO} \times \left(1 - \frac{AO}{12X} + \frac{3AO^2}{160X^2} - \frac{15AO^3}{2688X^3} + \frac{105AO^4}{55296X^4}\right), \&c.$$

Putting these into numbers as before, we shall find

O C = 49·835 or nearly 50·00.

SECTION III.

On the comparative strength of arches, and the method of finding the extrados of an arch from a given intrados.

From what has been observed at page 25, it may be inferred, that the strength of one part of an arch to that of another, will be proportional to the greatest weights those parts are capable of bearing; that is, as the cube of the secant of the curve's inclination to the horizon at those points, divided by the radii of curvature; and when two arches formed of similar materials have the same span and height, their comparative strengths at any two corresponding points will be also in the same proportion; but, since if one part of an arch fails, the whole will fall to ruin, and as the crown is the weakest part in all arches, it will only be necessary to compare them with each other at that point. Now as all curves at their vertices have no inclination to the horizon, the angles thereat will always be $0^{\circ} : 0\,0'$, and the secants become the

radii of those angles, and are the same or equal in every curve; whence it follows, that the comparative strength at the crowns, of two arches having the same span and height, is reciprocally as the radii of curvature at those points.

Taking for example an arch 100 feet span and 40 rise; the radii of curvature at the crowns in the different curves will be as follow:

	FEET.
Segment of a circle . .	51·25
Parabola	30·125
Ellipsis	62.5
Hyperbola	37.417
Catenary	36·9

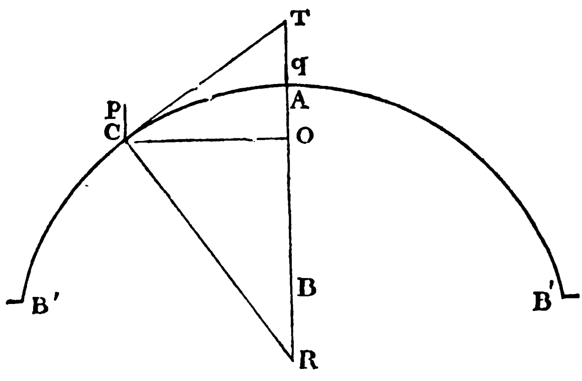
Hence, cæteris paribus, at the crowns of a parabola and an ellipse, the strength of the former to the latter will be as 62·5 to 30·125, &c. &c.

This will be more easily perceived if we consider the arches as polygonal, the sides being infinitely small. When the two first sides form a very large angle with each other, so that their mutual inclination approaches nearly to a straight line, the curvature is very small, and

they will evidently have less power to resist a pressure on the angle at the vertex, than when inclined to each other in a smaller angle, wherein the curvature is increased, and the radius of curvature consequently diminished. There are few, if any, instances of elliptical arches of large span in proportion to their height, in which the crowns have not sunk considerably.

PROPOSITION I.

To find the height of the superincumbent wall, or extrados, above every point of a circular arch, so that by its pressure all the parts of the arch, may be kept in equilibrio.



Let B′ A B′ be the segment of a circle, whose span is B′ B′, and height A B; R the centre, and A R the radius. Also let A q be the height of the wall at the vertex, and A O an abscissa to the ordinate O C at the point C, above which the height of the wall is to be found.

In figures, let . .	A q	=	1·00
	A B	=	40·00
	B′ B′	=	100·00
	A O	=	10.00
Consequently	A R	=	51·25

Now (see page 25).

$$\frac{(\text{sec. at vertex})^3}{\text{A R}} : \text{A q} :: \frac{(\text{sec.} \angle \text{T C O})^3}{\text{A R}} : \text{C p}.$$

Then by the nature of the circle, the radius of curvature is constant, and we shall have $\angle$ T C O* = 36°.24′, so that the analogy becomes,

* To find the angles which the line C T forms with C O, we must proceed as follows:

When the curve is circular (see figure to the proposition),

$$\text{C O} = \sqrt{2\,\text{A R} \times \text{O A} - \text{O A}^2}$$

$$\text{T O} = \frac{\text{O C}^2}{\text{O R}}$$

Then by plane trigonometry,

Since $\angle$ C O T is a right angle,

C O : radius :: T O : tang. $\angle$ T C O.

When the curve is parabolic (see figure to prop. 4).

$$\text{C O} = \sqrt{\frac{\text{A O} \times \text{B}'\,\text{B}^2}{\text{A B}}}$$

$$\text{O T} = 2\,\text{A O}$$

Then by trigonometry we have $\angle$ T C O as in last.

$(\sec. 0^\circ \cdot 00')^3$ or radius : 1 :: $(\sec. 36^\circ \cdot 24')^3$: C p $= 1 \cdot 9178$.

Then by taking different points in the curve, and proceeding as above, we shall have the different heights of the superincumbent wall at those points as required.

At the vertex A the secant is equal to the

When the curve is elliptical (see figure to prop. 2).

$$C\,O = \frac{\sqrt{B'\,B^2 \times \overline{B\,A^2 - B\,O^2}}}{B\,A^2}$$

$$O\,T = \frac{B\,A^2}{B\,O} - B\,O.$$

Whence we have $\angle$ T C O as before.

When the curve is hyperbolic (see figure to prop. 5).

$$C\,O = \frac{\sqrt{B'\,B^2 \times \overline{B\,O^2 - B\,A^2}}}{A\,B^2}$$

$$O\,T = B\,O - \frac{B\,A^2}{B\,O}.$$

Whence we have $\angle$ T C O as before.

When the curve is cycloidal (see figure to prop. 3).

By the properties of the cycloid (prop. 4, sect. 2),

$$\angle\, T\,C\,O = \angle\, A\,C'\,O.$$

Where A B is the diameter of the generating circle.

And $O\,C' = \sqrt{B\,O \times O\,A}$

And by plane trigonometry,

$O\,C'$: radius :: $A\,O$: tang. $\angle\, O\,C'\,A$ = tang. $\angle\, T\,C\,O$.

From prop. 5 in the last section, the reader may also find the $\angle$ T C O in the catenarean curve.

radius, and may be represented by C O; and the secant of the curve's inclination to the horizon at C, may be represented by T C, we shall therefore have the following analogy:

$$CO^3 : Aq :: TC^3 : Cp.$$

But because of the similarity of the triangles O R C, and T C O, O R and C R will be constantly proportional to C O and C T.

Consequently

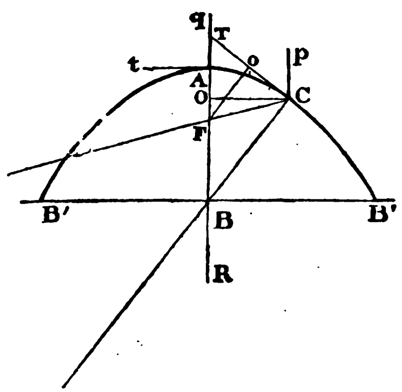
$$OR^3 : Aq :: CR^3 \; (= AR^3) : Cp;$$

that is $\dfrac{Aq \times AR^3}{OR^3} = Cp,$

which will be a general expression for the height C p at any point C of the curve; and is the same as Emerson makes it (see his Doctrine of Fluxions, London, 1743, page 248).

PROPOSITION II.

To determine the height of the extrados above every point of a parabolic arch.



Let B′ A B′ be a parabolic arch, whose span is B′ B′, and height A B.

Through F the focus of the parabola, draw C F, and from F let fall the perpendicular F o upon T C, then will the triangle F o C be similar to the triangle T O C; for the alternate angles O T C and T C p are equal, and the angle o C F is equal to the angle T C p. Consequently $\angle$ O T C $=$ $\angle$ o C F.

The angle T O C is also equal to the angle F o C, being both right angles.

Consequently F o may represent O C, the secant at the vertex, and F C may represent T C, the secant at the point C.

Therefore $F\,o^3$ and $F\,C^3$, are the cubes of the secants at those two points.

Again, the radius of curvature at C varies as $\frac{F\,C^2}{F\,o}$ (see Vince's Conic Sections, cor. to prop. 9), and when C arrives at A, F C and F o become each equal to F A.

Hence $\frac{F\,C^2}{F\,o} = F\,A.$

(See same work, prop. 8.)

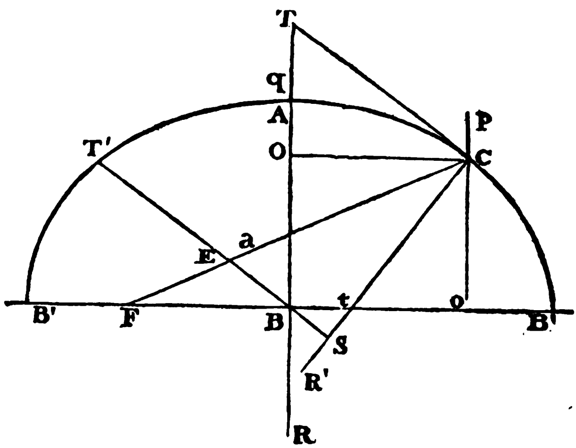
Therefore $\frac{F\,o^3}{\frac{F\,o^2}{F\,C}} : A\,q :: \frac{F\,C^3}{\frac{F\,C^2}{F\,o}} : C\,p;$

Or, $\frac{F\,o^3}{F\,o^3} : A\,q :: \frac{F\,C^3}{F\,C^3} : C\,p$, a ratio of equality.

Therefore C p is constantly equal to A q.

PROPOSITION III.

To determine the height of the extrados, above every point of an elliptical arch.



Let B′ A B′ be a semi-elliptical arch, whose span is B′ B′, and height A B.

Draw B T′ parallel to the tangent C T, and from the point C let fall C o perpendicular to B′ B′, and C S perpendicular to T′ B produced, if necessary, cutting B′ B′ in t.

Then because the angles T C S and O C o are right angles, and O C t is common to both,

take away the angle O C t from each, and the angle T C O is equal to the angle t C o. Also the angle C O T = angle C o t being both right angles; therefore the triangle C O T will be similar to the triangle C o t.

Consequently C o = B O may represent the radius or secant at the vertex, and $C\,t = \frac{A\,B^2}{C\,S}$, (see Vince's Conic Sections, prop. 15) the secant at the point C.

Therefore $B\,O^3$ and $\frac{A\,B^6}{C\,S^3}$ = the cubes of the secants at those two points.

From C to F, one of the foci of the ellipse, draw C F, cutting B T′ in E, and cut off C a equal to the chord of curvature passing through the focus (prop. 2, sect. 2), then will C S be to C E, as ½ C a is to the radius of curvature (see Vince's Conic Sections, prop. 21).

Now C E is always equal to B′ B, and when C arrives at A, C S becomes A B, also T′ B coincides with B′ B, and ½ C a or $\frac{T'\,B^2}{B\,B'}$, becomes equal thereto.

Therefore $A\,B : B'\,B :: B\,B' : \frac{B'\,B^2}{A\,B}$ the radius of curvature at A.

And $C\,S : B'\,B :: \frac{1}{2}\,C\,a \left(=\frac{B\,T'^2}{B\,B'}\right) : \frac{B\,T'^2}{C\,S}$ the radius of curvature at C (see above work, prop. 21).

But $\frac{B'\,B^2 \times A\,B^2}{C\,S^2} = T'\,B^2$ (same work, prop. 11. cor. 2);

therefore $\frac{T'\,B^2}{C\,S} = \frac{B'\,B^2 \times A\,B^2}{C\,S^3}$.

Consequently,

$$\frac{\frac{B\,O^3}{B'\,B^2}}{A\,B} : A\,q :: \frac{\frac{A\,B^6}{C\,S^3}}{\frac{B'\,B^2 \times A\,B^2}{C\,S^3}} : C\,p,$$

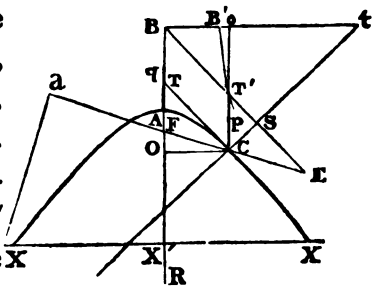
Or, $B\,O^3 : A\,q :: A\,B^3 : C\,p$;

That is, $C\,p = \frac{A\,q \times A\,B^3}{B\,O^3}$.

PROPOSITION IV.

To determine the height of the Extrados above every point of an Hyperbolic Arch.

Let X A X be an hyperbolic arch, whose span is X X, and height A X′. A B being the semi-transverse, and B B′ the semiconjugate axis.



From C draw C t perpendicular to C T, and produce B B′ to meet C t in t, and draw C o parallel to B O, then will the triangles T C O and o C t be similar, for the angles O C o and T C t are equal, being both right angles; take away from each the common angle T C o, there will remain the angle T C O equal to the angle o C t, and the angles T O C and t o C are both right angles. Therefore o C which is equal to B O, may represent the secant at the vertex, and C t the secant at C. Now as in the ellipse,

$$C\,t = \frac{A\,B^2}{C\,S}.$$

Therefore $B\,O^3$ and $\frac{A\,B^6}{C\,S^3}$ will be the cubes of the secants at those points.

Again through F, one of the foci of the curve draw F C, and produce it towards E and a, and cut off C a = the chord of curvature. Also from B draw B E parallel to T C, cutting C t in S.

Then (as in the ellipse) $C\,S : C\,E :: \frac{1}{2}\,C\,a$: radius of curvature.

Now C E = A B always, and when C arrives at A, C S also becomes equal to A B, and the half chord of curvature becomes equal to $\frac{B'\,B^2}{A\,B}$.

Therefore $A\,B : A\,B :: \frac{B'\,B^2}{AB} : \frac{B'\,B^2}{AB}$ the radius of curvature at A.

And $C\,S : A\,B :: \frac{1}{2}\,C\,a \left(=\frac{B\,T'^2}{A\,B}\right) : \frac{B\,T'^2}{C\,S}$ the radius of curvature at C.

But $B\,T'^2 = \frac{A\,B^2 \times B'\,B^2}{C\,S^2}$.

Therefore $\frac{A\,B^2 \times B'\,B^2}{C\,S^3} = \frac{B\,T'^2}{C\,S}$.

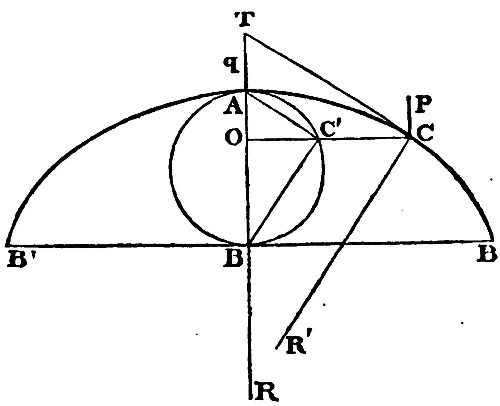
Consequently $\dfrac{\dfrac{B\,O^3}{B'B^2}}{A\,B} : A\,q :: \dfrac{\dfrac{A\,B^6}{C\,S^3}}{\dfrac{A\,B^2 \times B'B^2}{C\,S^3}} : C\,p$

Or $B\,O^3 : A\,q :: A\,B^3 : C\,p$;

That is $C\,p = \dfrac{A\,q \times A\,B^3}{B\,O^3}$.

PROPOSITION V.

To determine the height of the extrados above every point of a cycloidal arch.



Let B′ A B′ be a cycloidal arch, whose span is B′ B′, and height A B.

Then

$$\frac{(\text{sec. at vertex})^3}{A\,R} : A\,q :: \frac{(\text{sec.} \angle\, T\,C\,O)^3}{C\,R'} : C\,p;$$

$$\text{Or } \frac{O\,C^3}{A\,R} : A\,q :: \frac{T\,C^3}{C\,R'} : C\,p.$$

Draw A C′ parallel to C T, then by the nature of the cycloid, because T C is always parallel to A C, the triangle T C O will be similar to the triangle A C′ O, and consequently, to the triangle C′ B O, and their sides will be respectively proportional to each other; that is,

$$O\,C : T\,C :: B\,O : B\,C'.$$

And as the radii of curvature at A and C are respectively equal to 2 A B and 2 B C′, they are to one another as A B to B C′; that is as B C′ to B O. Consequently substituting these terms in the last proportion, we shall have

$$\frac{B\,O^3}{B\,C'} : A\,q :: \frac{B\,C'^3}{B\,O} : C\,p,$$

or by dividing the antecedent terms by $\frac{B\,O}{B\,C'}$,

we have $B\,O^2 : A\,q :: \frac{B\,C'^4}{B\,O^2} : C\,p.$

But $\frac{B\,C'^4}{B\,O^2} = B\,A^2$*,

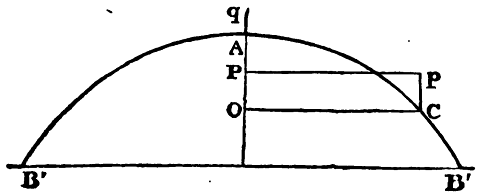
* For (Euclid Cor. Prop. 8. B. 6,) B C′ is a mean proportional between B A and B O, whence

Therefore $B O^2 : A q :: B A^2 : C p$, that is,

$$C p = \frac{A q \times B A^2}{B O^2}.$$

PROPOSITION VI.

To determine the height of the Extrados above every point of a Catenarian Arch.



The same analogy will obtain as in the other Propositions, the radius of curvature, &c. being found as directed in Sect. 2, Prop. 5.

Or let the tension of the curve at the vertex be called X.

Then (by the nature of the catenary) X will be constantly to A O,

As $X - A q$ to $q P$.

That is $X : A O :: X - A q : P q$.

$\frac{B C'^2}{B O} = A B$; and squaring both sides, we have

$\frac{B C'^4}{B O^2} = A B^2$.

Whence $P q = \frac{A O \times X - A O \times A q}{X}$.

But $C p = A O + A q - P q$.

Therefore

$C p = A O + A q - \frac{A O \times X - A O \times A q}{X}$, which being reduced becomes $C p = \frac{A q \times (X + A O)}{X}$.

OBSERVATIONS.

The subjoined table will shew at one view the height p C of the superincumbent wall, over any point C of the intrados; and by referring to the figures in plates 1 and 2, we shall see the general form of the extrados to each of the curves that have been treated of. In Emerson's Fluxions the reader will find, besides the curves already mentioned, the mode of determining the extrados to the logarithmic curve and cissoid; but as these curves would be seldom, if ever used in practice, it has been considered unnecessary to give them a place in this Treatise, which is designed to instruct the practical artist.

Curve.	*Value of* C p *at any point.*	
Circle.	$\frac{A\,q \times A\,R^3}{O\,R^3}$ or	The height at the crown multiplied by the cube of the radius, and the product divided by the cube of the vertical height of the point C from the horizontal diameter.
Parabola.	$p\,C = A\,q$ or	The vertical height of the extrados every where equal.
Ellipsis.	$\frac{A\,q \times A\,B^3}{O\,B^3}$ or	The height at the crown multiplied by the cube of the semiconjugate axis, and the product divided by the cube of the vertical height of the point from the transverse axis.
Hyperbola.	$\frac{A\,q \times A\,B^3}{(BA + OA)^3 \text{ or } BO^3}$ or	The height at the crown multiplied by the cube of the semitransverse axis, and the product divided by the cube (of the semitransverse axis added to the abscissa).
Cycloid.	$\frac{A\,q \times A\,B^2}{(A\,B - A\,O)^2}$ or	The height at the crown multiplied by the square of the diameter of the generating circle, and the product divided by the square (of the diameter aforesaid less the abscissa).
Catenary.	$\frac{A\,q \times (X + A\,O)}{X}$ or	The height at the crown multiplied into the sum of the tension at the vertex and the abscissa, and the product divided by the tension.

Thus it appears that all arches which spring perpendicularly, or whose secants at the springing are infinite, as the semi-circle, semi-ellipsis, and cycloid, will require to be loaded with an infinite weight over that point.

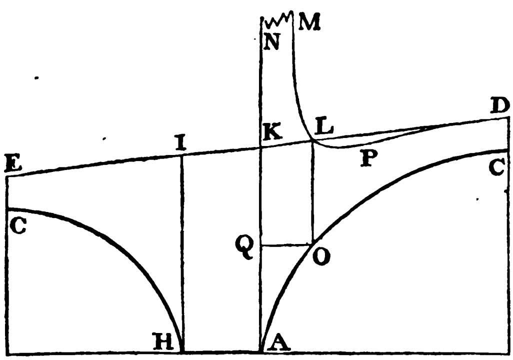
The extrados of a parabola will be another parabola, since p C is every where equal. And in the hyperbola the extrados continually approaches the intrados; but the scantiness of the haunches of these two curves, independent of their extreme meagre aspect, renders them unfit for the purposes of a bridge. Where great weights are required to be discharged (under particular circumstances) from the weakest parts of an edifice, as is often necessary in warehouses, and sometimes under apertures inverted, in order to bring an equal and uniform pressure on the foundations, the parabolic arch may with great propriety be introduced.

The catenarian curve will seldom be admissible, more especially in bridges; for where an horizontal roadway is required, the height at the crown must be very great. An arch of this kind of $31\frac{3}{10}$ feet span, and $15\frac{9}{10}$ feet rise, with an horizontal extrados, must be ten feet high at the crown (see fig. A, plate 2).

In general, however, by using segments of

the ellipsis, cycloid, and circle, we may obtain convenient roadways or extradosses, little differing from the conditions of the theory. In a segment of a circle, for instance, of 90 degrees, if the height at the crown be to its radius as 1 to $6\frac{1}{4}$, that is, about one ninth of the span, the roadway will be nearly horizontal (see fig. B, plate 2).

The numerous instances in which arches spring perpendicularly, without the infinite load which this theory requires over the springing point being given to them, seem to induce doubts of its truth; let us therefore shortly consider the apparent variance.



Take for instance two semiarches A C, C H, of a bridge (segments of circles); E I K L D

is the extrados likely to be applied in practice, in which the part K L D is the actual extrados over A O C; but the extrados which the theory requires is M L P D. Now D P L it will be seen by inspection nearly coincides with it as far as L; but the part of the extrados over A O appears deficient in the quantum of weight with which it ought to be loaded to obtain an equilibrium. But I H A O L, connected as it is by the bond of the stones of which it is composed, and the cement which unites them, may be considered as a solid mass of stone acting by its absolute gravity in a vertical direction upon A O Q, the part of the arch which requires the weight K L M N, and therefore may be considered as exerting a pressure equal to that of the infinite loading which theory requires. Thus in most cases the mode of constructing arches is not very far out of the conditions of the theory.

SECTION IV.

Of the Method of finding an Intrados to any given Extrados.

If the extrados or roadway over an arch be considered polygonal (as we considered the intrados in the first Section), and the direction of the first line of the intrados be given, we shall have an easy and tolerably correct method of finding the rest of the lines which compose the intrados, so as to equilibrate with the extrados; for the sides of both may be increased in number, so as to coalesce with the curves of which these lines are the chords (see pages 22 and 23, Sect. I.)

Let F l, l m, m n, n o′, o′ p′, p′ q′, (fig. 1, plate 3) be the sides of an extrados, and let C D be the direction of the first side of the intrados, cutting a vertical let fall from p′ in C, D q′ being the height at the vertex D.

Make the angles q D t, D q t, equal to the angle q D C, make D q equal to q′ D, and

draw the lines D t, q t, intersecting each other in t.

Produce (if necessary) C D indefinitely towards s, and in it take C s equal to q t, and in the vertical p′ C produced, make C p equal p′ C, and join p s.

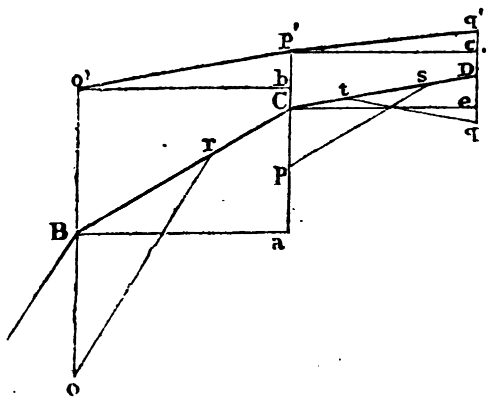
Through C parallel to p s draw C B, cutting a vertical from o′ in B, and produce it towards r, then making B r equal to p s, and joining o r, proceed as above for the heights under the points of the extrados n, m, l; o r is the direction of the side under n o′.

Through all the points thus found, as also those of the extrados, bend a flexible ruler and draw the curve lines, which will be similar to q G and D E.

The operations for finding an intrados to any other extrados, will be precisely the same; as may be seen fig. 2, plate 3, where the extrados is horizontal.

This construction is evidently the converse of that in Prop. 6, Sect. 1; and the demonstration of both depends upon the lemma to Prop. 5. Sect. 1; for in both it is supposed that the arch consists of an assemblage of beams kept in equilibrio by weights placed at the angular points.

If it be required to find, arithmetically, the heights of the several lines p′ C, o′ B, &c. the following is an easy process.



Let $q'D = 1{\cdot}00$
$o'p' = 4{\cdot}90$
$p'q' = 4{\cdot}25$
$\angle\ p'q'D = 85^\circ{\cdot}00'$
$\angle\ o'p'C = 80^\circ{\cdot}00$
$\angle\ CDq = 80^\circ{\cdot}00$

Having let fall the verticals q′ D, p′ C, o′ B, draw p′ c, C e, and o′ b, at right angles thereto.

Then by plane trigonometry,

$$p'c = \frac{p'q' \times s.\ \angle\ p'q'c}{s. \angle\ p'cq'} = 4{\cdot}233$$

$$e\,D = \frac{C\,e\,(=p'\,c) \times s.\angle\,D\,C\,e}{s.\angle\,C\,D\,q} = 0{\cdot}7465$$

$$c\,q' = \frac{p'\,c \times s.\angle\,q'\,p'\,c}{s.\angle\,p'\,q'\,c} = 0{\cdot}3703$$

$$c\,e = p'C = p\,C = D\,q' + D\,e - q'c = 1{\cdot}0 + 0{\cdot}7465 - 0{\cdot}3703 = 1{\cdot}3762.$$

Now

$$D\,t = \frac{D\,q \times s.\angle\,C\,D\,q}{s.\text{ of }(180^\circ - 2\angle\,C\,D\,q)} = 2.879\ (= C\,s).$$

Also

C p + C s : C s − C p :: tangent ½ sum of the angles C p s, C s p : tangent ½ difference of those angles.

So that $\angle\ C\,p\,s = 56^\circ{\cdot}31'$
$\angle\ C\,s\,p = 23^\circ{\cdot}29'$

$$o'\,b = \frac{o'\,p' \times s.\angle\,o'\,p'\,b}{s.\angle\,o'\,b\,p'} = 4{\cdot}823\ (= B\,a)$$

$$p'\,b = \frac{o'\,p' \times s.\angle\,p'\,o'\,b}{s.\angle\,o'\,b\,p'} = 0{\cdot}7941.$$

As the angle B C a is equal to the angle C p s, draw B a parallel to o′ b, then B a C is a right angle; consequently $\angle$ a B C $= 90^\circ - 56^\circ{\cdot}31' = 33^\circ{\cdot}29'$,

And

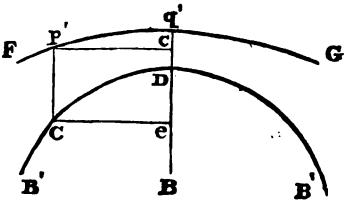
$$C\,a = \frac{B\,a\,(= o'\,b) \times s. \angle\, a\,B\,C}{s. \angle\, B\,C\,a} = 3{\cdot}192$$

$$o'\,B\,(= b\,a) = C\,p' + C\,a - b\,p' = 1{\cdot}376 + 3{\cdot}192 - {\cdot}7941 = 3{\cdot}774.$$

The whole extent is B a + C e (= p′ c) = 4·823 + 4·233 = 9·056, and height = C a + e D = 3·192 + 0·7465 = 3·9385.

Many intradosses may be found by the above method to a single given extrados, the span of each depending upon the degree of inclination given to the first line D C; that is, if the angle q′ D C were only 92°.00′, the intrados will be of less height or rise in proportion to any given span, than if the said angle were 95°·00′. Thus the architect has the means of choosing the most convenient and pleasing form for his intrados as circumstances may require*.

* If, having an extrados of a given curve, it be required to adapt an intrados thereto, whose span and height must be of certain given dimensions, the following operation will be necessary;



Let F p′ q′ G be any extrados, as a segment of a circle, ellipse, &c. &c. whose ordinate p′ c, and abscissa q′ c, are known at every point thereof.

If, instead of treating the arch as an assemblage of beams with weights at the angles, we

Let q′ D (height at crown)	=	7·00
q′ c the abscissa	=	3·00
C e, and p′ c its common ordinates (supposing, in the present example, the extrados to be the segment of a circle whose radius is 230) will	=	34·00
B B′ the half span	=	50·00
D B the height or rise	=	40·00

To find D c.

Put C e (= p′ c) = y
B B′ .... = s
D B .... = h
D e .... = x

I. $p'\,C = e\,c = C \times \dfrac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^2} = \dfrac{C}{\dot{y}} \times$ fluxion of $\dfrac{\dot{x}}{\dot{y}}$.

It is evident $e\,c = q'\,D + D\,e - q'\,e = \dfrac{C}{\dot{y}} \times$ fluxion of $\dfrac{\dot{x}}{\dot{y}}$.

II. Assuming $\dot{y} = \dfrac{\dot{x}}{z}$ we have $z = \dfrac{\dot{x}}{\dot{y}}$ and $\dfrac{C}{\dot{y}}$ × fluxion of $\dfrac{\dot{x}}{\dot{y}} = \dfrac{C\,z\,\dot{z}}{\dot{x}}$

That is $7 + x - 3 = \dfrac{C\,z\,\dot{z}}{\dot{x}}$,

And $x + 4 = \dfrac{C\,z\,\dot{z}}{\dot{x}}$, or $x\,\dot{x} + 4\,\dot{x} = C\,z\,\dot{z}$.

The fluent of this expression is $\dfrac{x^2}{2} + 4\,x = \dfrac{C\,z^2}{2}$,

were to consider it as covered by a loading diffused over it, it is evident, from the observations

$$\text{or } 2x^2 + 16x = 2\,C\,z^2, \text{ or } x^2 + 8x = C\,z^2;$$

$$\text{from which } z = \sqrt{\frac{x^2 + 8x}{C}}.$$

III. Now as $\dot{y} = \frac{\dot{x}}{z}$,

$$\dot{y} = \dot{x} \div \sqrt{\frac{x^2 + 8x}{C}} = \dot{x}\sqrt{\frac{C}{x^2 + 8x}} = \dot{x}\sqrt{\frac{C}{x^2 + 2 \times 4x}}.$$

The fluents of which are $y = \sqrt{C} \times$ * hyperbolic logarithm

of $(x + 4 + \sqrt{x^2 + 8x})$

but at the vertex where $x = o$ we have

$y = \sqrt{C} \times$ hyperbolic logarithm of 4;

So that the corrected fluent becomes $y = \sqrt{C} \times$ hyp. log. of $\frac{x + 4 + \sqrt{x^2 + 8x}}{4}$.

IV. To obtain the constant quantity C. When e arrives at B,

we have $x =$ D B which call h,

$y =$ B′ B whieh call s,

and the abscissa of the extrados $= 5 \cdot 5$.

Then repeating the above operation, we obtain

$$s = \sqrt{C} \times \text{hyp. log. of } \frac{h + 1\cdot5 + \sqrt{h^2 + 3h}}{1\cdot5},$$

$$\text{Consequently } \sqrt{C} = s \div \text{hyp. log. of } \frac{h + 1\cdot5 + \sqrt{h^2 + 3h}}{1\cdot5}.$$

V. Hence we obtain,

* The hyperbolic logarithm of any number may be found by multiplying its common or Briggs's logarithm by 2·302585, &c. the product being the hyperbolic logarithm of the number required.

at the end of Section I. that a nearer approximation to the true form of the intrados would be

$$y = \frac{s \times \text{hyp. log.} \dfrac{x + 4 + \sqrt{x^2 + 8x}}{4}}{\text{hyp. log.} \dfrac{h + 1{\cdot}5 + \sqrt{h^2 + 3h}}{1{\cdot}5}}$$

$$\text{or } \frac{y}{s} = \frac{\text{hyp. log.} \dfrac{x + 4 + \sqrt{x^2 + 8x}}{4}}{\text{hyp. log.} \dfrac{h + 1{\cdot}5 + \sqrt{h^2 + 3h}}{1{\cdot}5}}$$

$$\text{Hence } \frac{y}{s} \times \text{h.l.} \frac{h + 1{\cdot}5 + \sqrt{h^2 + 3h}}{1{\cdot}5} = \text{h.l.} \frac{x + 4 + \sqrt{x^2 + 8x}}{4}.$$

Suppose the last expression $\frac{y}{s} \times \text{h.l.} \frac{h + 1{\cdot}5 + \sqrt{h^2 + 3h}}{1{\cdot}5}$ when put into numbers to be a hyperbolic logarithm, and find the natural number corresponding to it, which call N;

$$\text{then } N = \frac{x + 4 + \sqrt{x^2 + 8x}}{4},$$

$$\text{or } 4N = x + 4 + \sqrt{x^2 + 8x},$$

$$\text{and } 4N - x - 4 = \sqrt{x^2 + 8x}.$$

Squaring both sides, we have

$$16N^2 - 8Nx - 32N + x^2 + 8x + 16 = x^2 + 8x,$$

$$\text{or } 16N^2 - 32N + 16 = 8Nx;$$

$$\text{whence } x = \frac{2N^2 - 4N + 2}{N}.$$

It is only in the second and third steps that a change is necessary; namely, where the value of the abscissa of the extrados varies according to the circumstances of the data. Any

obtained by multiplying the heights D q, C p′, B o′, &c. found as above, by radius and by the secants of the angles D C e, C B a, &c. respectively.

person the least acquainted with common algebra may carry this operation through.

In numbers as follows:

$\frac{y}{s} = \frac{34}{50} = \cdot 68$, and as $h = 40$, the expression $\frac{h + 1\cdot 5 + \sqrt{h^2 + 3h}}{1\cdot 5}$ becomes $\frac{40 + 1\cdot 5 + \sqrt{1600 + 120}}{1\cdot 5} = 55\cdot 31$, the hyperbolic logarithm of which is 4·01297, this being multiplied by ·68 gives 2·72884, and looking into a table of hyp. logs. we have the natural number corresponding to this = 13·13, which we call N;

then $x = \frac{2\,N^2 - 4\,N + 2}{N} = \frac{344.8 - 52\cdot 52 + 2}{13\cdot 13} = 22\cdot 4$ for the length of the abscissa D e, whence the value of c e, that is of p′ C, is known.

For an horizontal extrados (see Hutton's Principles of Bridges, sect. 3.) the equation to the curve of the intrados is found to be

$$y = s \times \frac{\text{hyp. log. of} \frac{a + x + \sqrt{2\,a\,x + x^2}}{a}}{\text{hyp. log. of} \frac{a + h + \sqrt{2\,a\,h + h^2}}{a}}$$ where a expresses q′ D, the height at the crown.

CONCLUDING OBSERVATIONS ON THE EQUILIBRIUM OF ARCHES.

It has long been a subject of complaint, that the theories of the statical equilibrium of bodies afford little assistance in the execution of any works where it is of importance to obtain a requisite degree of stability with the least possible expense of materials. Yet it can hardly with justice be considered as a reproach to mathematicians that they have not been able to discover theories which should comprehend all the relations of the subjects involved, and which should be applicable to the actual circumstances of every case, since this would require a more intimate knowledge of the laws of mechanical action than we at this time possess.

Under the disadvantages arising from our imperfect acquaintance with the properties of bodies, the only thing which can be done is to assume the absolute perfection of all the qualities of the materials employed, to investigate the conditions of equilibrium according to that supposition, and then, to leave to the persons who are to apply the conditions practically,

the care of making such modifications in them as are indicated by the results of the best experiments that have been tried.

The objections that are made to the theory of arches just delivered are, that the voussoirs are supposed indefinitely thin, and that the loading above is supposed to act only in the vertical direction. Now, with respect to the first, the smallest excess of pressure on any part of the arch, above what the theory assigns, would force the voussoirs from their places; and with respect to the second, when the loading consists of loose material, as rubble or gravel, it exerts pressures laterally as well as vertically.

The latter point may be dispatched in a few words; the principal part of that lateral pressure is counteracted by the resistance of the piers, and the remainder is too inconsiderable to deserve notice. It may also be observed that, of late, the practice of filling up the haunches with loose material seems to be abandoned, and that of solid walls, parallel to the length of the bridge, to be adopted. This work not only presses vertically upon the arch, but, by its cohesion renders an excess of weight at the crown less likely to force up the arch at the haunches.

The weakness arising from the other circumstance would be of serious consequence, if it were not that the artists have invariably given to the voussoirs a certain degree of thickness, in order that the divergency of the joints may be sufficient to allow them to keep their places. This is certainly an indispensable requisite; and as the tendencies of these stones downward upon their joints produce pressures, which are different from those produced by the vertical loading in the catenarian theory, it may be expected that attention should be paid to something more than the equilibrium produced by such vertical loading.

The simplest method of adapting the catenarian theory to practice, seems to be that of making the courses of voussoirs balance themselves, as in Mr. Atwood's theory, viz. by equalising the tendency which any given course has to rise, by a resultant of the lateral pressure of the superior courses, with the tendency downward in an opposite direction by a resultant of the weight of the given course. The weights of the voussoirs, and consequently their lengths being determined by such means, we may consider the extrados of these voussoirs as a new intrados, and proceed to determine a new extrados

for the roadway by the heights of the columns of vertical loading, according to the catenarian theory: and as the exterior curve of the voussoirs will always be more flat than the interior curve, and will always rise at an acute angle with the horizon, it is manifest that the final extrados will approximate nearer to a right line, and will consequently be more convenient for a roadway than that which would be obtained either by the catenarian theory in its simple state, or by the method of equilibrial voussoirs.

Let it also be recollected, that the voussoirs or stones of which the arch is composed, have flat surfaces in contact with each other, and that the cement and friction add to their stability. It is true that no theory will shew what additional weight (in practice) an arch can sustain on its weakest part; but an arch of equilibration is certainly better able to sustain a greater weight accidentally placed on any part than another arch would be.

Strict mathematical precision cannot always be obtained; but by attention to what has been shewn, the architect will be better able to combine the requisite strength with that beauty which it will ever remain in his province alone to impart to the design.

Of the thickness given to the voussoirs at the crown, there have been many differing examples; in general, we may put it down from $\frac{1}{14}$ to $\frac{1}{17}$ of the span of the arch, varying according to particular circumstances. Palladio gives the thickness of them in the ancient* bridge at Rimini, at $\frac{1}{10}$ of the span to the middle arch, and $\frac{1}{8}$ to the side arches.

Again in the bridge at Vicenza, he gives the voussoirs to the middle arch $\frac{1}{12}$ of the span, and the side arches $\frac{1}{9}$. But in a design by himself†, he makes the thickness of the voussoirs in the middle arch $\frac{1}{17}$, and that of those of the side arches $\frac{1}{14}$ of the span.

L. B. Alberti recommends‡ them to be of the largest and hardest stones; and directs no stone to be used that is not at least $\frac{1}{10}$ of the chord of the arch; nor (says Alberti) should the chord itself be longer than six times the thickness of the pier, nor shorter than four times.

In Westminster bridge the voussoirs are about $\frac{1}{15}$ of the span, and at Blackfriar's bridge nearly the same.

* —— il lor modeno e per la decima parte della luce de' maggiori e per l' ottava parte della luce de' minori. Cap. 11. lib. 3.

† Lib. 3. cap. 14. ‡ Lib. 4. cap. 6.

It is to be observed that in most of, if not in all, their bridges, the ancients did not increase the dimensions of their voussoirs from the crown towards the springing or coussinet, but made them of an equal thickness throughout; in this they were followed by Palladio, and all the Italian architects. They sometimes, however, made the alternate voussoirs larger than the others, as Mr. Labelye has done in Westminster bridge.

While the voussoirs are considered as indefinitely short, and are held in a state of tottering equilibrium by the vertical pressure of the superincumbent loading alone, as the theory, in its simple state requires, the arch would not be calculated to support any extraneous weight. In practice, the voussoirs are of considerable length, and their adjacent surfaces are in contact; and when an additional weight is brought to act upon any part, suppose at the crown, it will cause the joints at such part to open on the concave side; the haunches will consequently be forced up, and their joints will open on the convex sides: but while the imaginary lines, expressing the directions of pressure passing through the voussoirs, are not so much distorted as to be thrown out of the limits of the

surfaces of the blocks, the arch will stand, though loaded at the crown with a certain degree of weight beyond what the strictness of the theory allows; when these imaginary lines are removed out of the limits of the adjacent blocks or voussoirs, the arch will be completely destroyed. It therefore follows that the voussoirs should be as large as may conveniently be got; the larger they are the more may the distortion be increased, without endangering the structure, since the directions of their pressures will be less likely to exceed the limits of their magnitude.

In general, while a line can be drawn from the crown to the haunches, passing entirely within the surfaces of the voussoirs, the arch will stand; but, when any part of the line falls out of their surfaces, the stability of the arch is instantly destroyed.

Enough, it is hoped, has been said to convince the artist of the necessity of balancing his arch with caution, and as exactly as circumstances will allow; and where an equilibrium according to this theory cannot be obtained, the fertile mind of an ingenious artist will naturally furnish expedients to obviate the inconvenience likely to arise from a want of it.

SECTION V.

On the horizontal drift or shoot of an arch, and the thickness of the piers.

PRELIMINARY OBSERVATIONS.

If a bridge should consist of but one arch between the abutments, and each abutment does not immediately rest against an immoveable object, such as the bank of a river, it will be evident, if the materials composing the arch have a tendency to yield to any pressure in the direction of the length of the bridge, that the effect of this lateral thrust will be either to push the piers off horizontally, or to overturn them. The former effect may take place if the curve of the arch should begin near the foundations; the latter, if it rest upon piers considerably elevated. In either case the thrust must be counteracted, by giving a proper thickness to the abutments.

If a bridge consist of several arches, and we choose to consider the lateral thrust of each arch alone, without regard to the contrary

thrusts of those with which it is connected, the counteraction must be effected by the same means.

No theory, purely mathematical, has yet been discovered for determining the equilibration of an arch in this respect, nor have all the circumstances attending this species of pressure been satisfactorily ascertained. The principle which has been generally assumed as the basis of the investigation is, that the materials of which each semi-arch as A C D B, (see the figures to Plate 1) is composed, have a freedom of motion and a tendency to roll or slide over the intrados B D, by a resultant of the action of gravity, and that they are retained by the side B A of the wall or pier. The equivalent of all these forces being estimated in some particular direction, is considered as the force which tends to push off or overturn the pier or abutment.

To determine this force, it becomes necessary to find the centre of gravity of any longitudinal section of a semi-arch; for in that point, by the laws of mechanics, the mass of materials may be considered as collected. Or, if the force is to be estimated in a horizontal direction, it will be merely necessary to determine the situation of a vertical line passing through the centre of

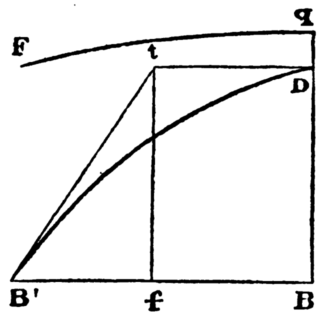
gravity, which is more easy than to determine that of the centre of gravity itself.

If the arch be perfectly equilibrated by its loading, according to the rules delivered in this treatise, and if only such a portion of the arch is employed as will permit its extrados to serve accurately for the intended roadway, then the two following propositions will shew how, in a very easy manner, to determine the situation of the vertical passing through the centre of gravity, and how from thence to determine the lateral thrust exerted by the arch.

PROPOSITION I.

PROBLEM.

To find the horizontal distance B′ f *from the abutment or springing* B′, *of the centre of gravity of the materials with which the equilibrial arch* B′ D *is loaded.*



At B′ and D draw the tangents B′ t, D t, intersecting each other in t. The centre of gravity will be in a vertical line passing through t. From the point t of intersection let fall the vertical t f, cutting B′ B in f. Then B′ f is the horizontal distance of the centre of gravity of the semi-arch B′ F q D, from the abutment B′, and f B the horizontal distance from q B.

For the arch line B′ D with its loading may be considered as resting on the points B′ and D,

and exerting a pressure there in the directions of the tangents t B′, t D; the re-actions of the materials which support the half arch at those points are consequently to be considered as forces in the opposite directions B′ t, D t. But in mechanics, where a body is sustained by two forces, those forces produced will meet, either in the centre of gravity of the body supported, or the centre of gravity will be in a vertical line passing through that point of intersection.

EXAMPLE.

$$\begin{aligned}
\text{If } \angle\, f B' t &= 84°\cdot 46' \\
\text{and } \angle\, B' f t &= 90°\cdot 00' \\
\angle\, B' t f &= 5°\cdot 14' \\
\text{Then } t f = B D \text{ being} &= 40°\cdot 00' \\
\text{and } \; B B &= 50°\cdot 00'
\end{aligned}$$

By plane trigonometry,

We shall have

$$B' f = \frac{D B \times s.\ \angle\ B' t f}{s.\ \angle\ f B' t} = 3\cdot 699$$

$$\text{and } f B = 40 - 3\cdot 699 = 36\cdot 301.$$

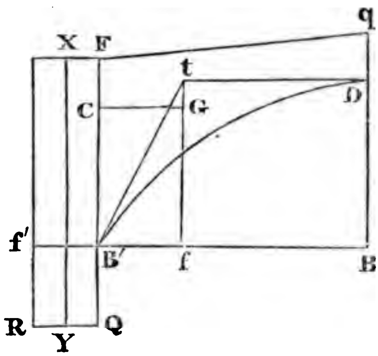
When the curve thus loaded to equilibrium is one of those mentioned in note to Prop. 1,

Sect. 3, the angle f B′ t may be found by the means there shewn; but for any other curve than those, other means to find that angle must be used.

PROPOSITION II.

PROBLEM.

To find the horizontal shoot or drift of any semi-arch B′ D *loaded to equilibrium (that is, the force it exerts in an horizontal direction at* B′).



Let B′ F q D be the semi-arch. Find (by the last prop.) the horizontal distance B′ f of the centre of gravity of the arch loaded to equilibrium and draw f t perpendicular to B′ B.

Find the area of B′ F q D which may represent the weight of the semi-arch, since the weight is proportional to the area of the section. Then by mechanics as

t f : B′ f :: weight of the semi-arch : its shoot or horizontal drift in the direction f B′.

OBSERVATIONS.

The method of estimating the lateral thrust, contained in the two preceding propositions, must not be employed when the arch rises perpendicularly, or nearly so to the horizon. In fact, the intersection t of the tangents D t, B′ t would then take place upon, or very near to the vertical line B′ F, and the semi-arch would appear to have no lateral thrust, a circumstance which, though true in theory, must not be admitted in practice. For as the extrados cannot, on account of its great height over the springing courses, serve for a road, it is evident that by cutting off the part above the line of the intended road, the position of the centre of gravity is altered, and to determine the vertical line t f, the centre of gravity of the semi-arch must be found by one of the usual methods. The easiest

would undoubtedly be to cut the form of the half arch in pasteboard, and make it support itself upon a point, the point thus found will be the place of the centre of gravity, which suppose to be at G: then the horizontal shoot or drift may be found as in Prop. 2, of this Section.

Now to determine the thickness of a pier necessary to counteract the horizontal thrust of the arch, we must consider that this thrust is to be resisted by the friction which the stones composing the pier experience in sliding upon each other. From sundry experiments it has been found, that in some kind of stone the friction of one block moving horizontally upon another, is equal to one third of the weight of the moving block. If we adopt this determination, the weight of the pier ought to be equal to three times the horizontal drift of the arch to produce an equilibrium.

If A = the area of the semi-arch, which area may represent its weight: then by Prop. 2, we have $\frac{B' f \times A}{t f}$ = the horizontal thrust; hence

$\frac{3 B' f \times A}{t f}$ = the area of the vertical section F R of the pier, which area may represent the weight

of the pier, and $\frac{3\,B'f \times A}{t\,f \times X\,Y}$ = the thickness R Q required.

The specific gravity of the materials composing the arch and pier, is here supposed to be the same, which is generally the case, and on this account, it does not enter into the formula. The height X Y is given, and we suppose the whole pier to stand dry, or out of the water.

If we could determine the conditions of equilibrium between the stability of the pier and the effort of the arch to overturn it about R, we might estimate the horizontal thrust as before; thus let G be the centre of gravity of the half arch; draw G C parallel to B′ B, and suppose the horizontal thrust to be applied at C, at the extremity of the arm C Q of the bent lever C Q R. In this case $\frac{B'f \times A \times C\,Q}{t\,f}$ will express the effort of the arch to overturn the pier. But F Q × R Q is equal to the area of the section of the pier, and represents its weight; and this weight is supposed to act in the vertical line X Y, passing through the centre of gravity of the pier, it consequently acts at the extremity of the lever R Y, which is equal to $\frac{1}{2}$ R Q.

Therefore we have $F\,Q \times \frac{1}{2}\,R\,Q^2$ to express the stability of the pier. Then equating these two forces and reducing, we have

$$R\,Q = \sqrt{\frac{2\,B'f \times A \times C\,Q}{t\,f \times F\,Q}}$$ for the

thickness required, supposing as before, the whole pier to stand out of water.

But inasmuch as part of the pier is in most cases immersed in water, it thereby loses so much of its weight as is equal to the weight of a quantity of water, whose bulk is equal to that of the immersed part of the pier. To allow for this, let us suppose the material of the bridge to be granite, the specific gravity of which, to that of water is as $3\frac{1}{2}$ to 1, and let the pier stand in water as high as the springing B′. Then the stability of the pier will be expressed by $1{\cdot}75\,F\,Q \times R\,Q^2 - 0{\cdot}5\,B'\,Q \times R\,Q^2$, and the effort of the arch to overturn it, will be $\frac{3{\cdot}5\,B'f \times C\,Q \times A}{t\,f}$; equating these terms,

we have $R\,Q = \sqrt{\frac{7\,B'f \times C\,Q \times A}{t\,f\,(3{\cdot}5\,F\,Q - B'\,Q.)}}$.

The thickness of the piers determined on this principle will be found rather greater than is allowed by modern architects, and to reduce the results of theory nearer to the general practice,

mathematicians have brought the point of application of the thrust of the arch nearer to the foot than the point C is; but as this is rather arbitrary, it has been thought better to leave it in a horizontal line passing through G. It is for the architect to exercise his judgment in making such modifications of the theory.

It is advisable (if possible) to construct the pier of heavier materials than the arch, as by that means it will not require so great a thickness, and a greater waterway * will consequently be obtained.

If the pier be constructed as we have supposed above, that is, independently of the resistance arising from the abutments, or from the neighbouring arches, a considerable saving of expense in the centering of a bridge, where several arches are required, will accrue; for, allowing the piers to be individually capable of

* In the Philosophical Transactions 1758, is a paper by Robertson, on the fall of water under bridges, wherein the following formula is given.

$$\left(\left(\frac{25\,b}{31\,c}\right)^2 - 1\right) . \frac{v\,v}{4\,a} = \text{fall of the river occasioned by the piers.}$$

Where b = the breadth of the river,
c = the waterway,
v = the velocity of the river in one second,
a = the fall of a heavy body in one second = 16·0899.

resisting the drift, the centre of one arch may be struck before another is begun, and it will serve for its opposite and corresponding one, that is, supposing them to be of equal sizes, which is, unless under particular circumstances, always the case. Otherwise the arches depend, for their stability, upon the counteracting efforts of each other, and additional centering, one of the most expensive pieces of machinery used in their construction must be necessarily employed.

The Italian as well as the ancient architects seem to have regulated the thickness given to the piers of bridges, by proportioning them to the span of the arches, without regard to any other circumstances.

In the little bridge over the Ilyssus, near Athens, the middle arch of which is only 19 feet 10 inches span, the extraordinary thickness of nearly 8 feet 6 inches is given to the piers, being nearly five twelfths of the span.

In the bridge at Rimini, admired by Palladio for its beauty* as well as strength, nearly the same proportion is observed †.

* "Mi pare il piu bello e il piu degno di considerazione, si per la fortezza, come per il suo compartimento."

† "I pilastri sono grossi poco meno della mettà della luce degli archi maggiori." Lib. 3, cap. 11.

In the bridge over the Bacchilione, at Vicenza, the piers are one-sixth of the span of the middle arch; and in a design by Palladio himself, he makes the pier to the middle arch one-fifth of the span.

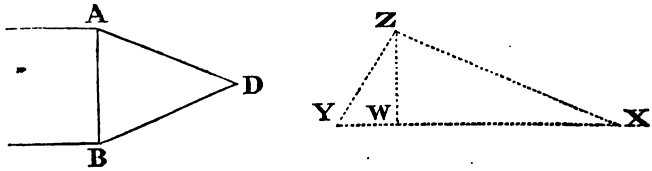
L. B. Alberti says, that the piers ought not to be less than one-sixth, nor morethan one fourth of the span.

At Westminster bridge the piers to the middle arch are a little more than one-fourth of the span, and at Blackfriars bridge a little more than one-fifth.

It is evident, however, after what has been shewn, that in order to estimate the proper thickness, the circumstances of the span, height, forms of the extrados and intrados, must enter into the consideration.

The extremities of the piers of bridges are usually terminated in points towards the current of a river, in order to diminish the pressure of the water against the piers, and lessen the eddy which the water, flowing off laterally after striking the piers, forms at the shoulders. Now, to compare the pressure sustained by a pier terminating in a head, as A B, at right angles with the stream, with that sustained by one whose

extremity forms an isosceles triangle as A D B on the plan: let X Y



parallel to the length of the pier represent the force with which the water would strike any point of the square head A B: this force may be resolved into X Z, which is parallel and Z Y which is perpendicular to the face A D. The portion of the force of water represented by X Z being parallel to A D produces no effect upon it, and the remaining force Z Y may be resolved into the two Z W perpendicular to, and W Y coincident with the length of the pier: the portion of the force of the water represented by Z W is counteracted by an equal force in an opposite direction, and there only remains the force represented by Y W tending to push the pier from its place, consequently the pressure of the current against a pier terminated perpendicularly to the stream, as A B is to the pressure against one terminated by the triangle A D B, as X Y to W Y.

If the angle at D were a right angle, the force against A D B would be only half the force against A B, and it is evident that the force must become less in proportion as the angle becomes more acute; and in fact it may be shewn that the force is inversely proportional to the square of the side A D.

It is evident too, that curvilinear piers oppose less resistance to the stream the nearer their faces approach to right lines; but care should be taken not to make the angle at D too acute, on account of the injury which vessels may sustain by being driven against it, and the eddy which is produced when the current sets obliquely to the pier.

Though some objections have been made to this theory, nothing better has yet been submitted to calculation.

OF DOME VAULTING.

A work of this kind would be incomplete if it did not contain some account of the principles of equilibrium in domes of masonry; it is therefore purposed to conclude this section by a few words on that species of building.

Let Figure 1, Plate 4, represent part of a vertical section through the centre of the dome; let A B represent one half of the keystone, and the line A B the direction in which it presses against the adjacent stone B C, that is perpendicular to the joint B: find g the centre of gravity of the half keystone; draw the vertical line g v, and make it of any length at pleasure to represent the weight of the half-stone A B, and through v draw v f at right angles to g v, meeting A B produced in f. If g v represent the weight of the stone A B, then will g f represent its pressure against the adjacent stone B C, in a direction perpendicular to the joint B. Now make B b equal to and in the same direction as g f, and on an indefinite vertical line drawn through B take B c to represent the weight of the stone B C, complete the parallelogram c b, and draw the diagonal B d; then will B d represent the quantity and direction of the pressure of the two stones A B, B C upon the joint C. Again, make C e equal to and in the same direction as B d, and on an indefinite vertical line through C take C h to represent the weight of the stone C D; complete the parallelogram h e, and draw the diagonal C k, then will C k represent the quantity and direction of the

pressure of the three stones A B, B C, C D upon the joint D. Make D m = C k, and proceed as before.

It is evident from this construction that the pressure of the materials in the vertical sections of the dome changes its direction continually: viz. from A B to B C, to C D, and so on; and, if the breadths A B, B C, &c. of the stones were indefinitely small, the polygon A B, B C, C D, &c. would become a curve concave towards the axis of the dome.

Now, in forming a cylindrical vault which should support itself without any loading, the above construction might be employed, and in that case, the several verticals B c, C h, D o, &c. which represent the weights of the arch stones, or the vertical pressures which they exert upon the next lower joints C, D, E, &c. must be equal to each other when the lengths of the arch stones are equal, which is here supposed, and each must be equal to twice g v (g v representing the weight of a half voussoir); consequently the polygon formed would be an approximation to the common catenary. But in forming a dome, the keystone at A rests on a circular base formed by the next inferior horizontal course, and the lower circumference of each

horizontal course is greater than the upper. Therefore each horizontal course rests upon a greater number of points than the course next superior to it, and the number of points which the courses rest upon is proportional to the circumferences, or to the radii of their bases. Hence, if we have assumed any line, as g v, to represent the weight of the half arch-stone A B, or rather to represent the vertical pressure which that half stone exerts on the oblique joint B; then it is plain that to get the weight of the arch-stone B C, or rather the vertical pressure it exerts against the oblique joint C, we must make B c equal to 2 g v diminished in the proportion of the radius of the horizontal course B C taken at C, to the radius of the same course, taken at B; that is z C : w B :: 2 g v : B c. Afterwards C h instead of being made equal to B c, must be obtained by the following proportion x D : z C :: B c : C h; and in the same manner the other verticals may be found.

It must be observed, however, that the radii z C, x D, &c. cannot be accurately known till the positions of the courses B C, C D, &c. are determined, but an approximation may be made to these radii by first making B c = 2 g v determining the direction of the diagonal B d, and

the position of C as described above. The radius z C may then be employed to get a more correct value of B c, from which the position of C may be determined with sufficient accuracy. In the same manner the approximate value of x D may be found, and from thence the more correct place of the point D, and so on.

Now, if the dome were so constructed that the several lines of direction A B, B C, &c. were always within the thickness of the voussoirs, it would stand and be in perfect equilibration; but if the lines of direction of the forces fall on the exterior of the curve of the dome, it is evident that the pressures of the upper courses would give the lower ones a greater tendency to slide outward at the joints than would be counteracted by their weights, and consequently the dome would fall.

Again, if the directions of the forces fell on the interior of the curve, the stones would tend to fall inward, but as this would take place equally in every stone of each horizontal course respectively, the tendency would not be obeyed, the breadth of the stones on the exterior being greater than on the interior. It is even considered that in proportion to the flatness of the dome, the action of gravity binds the stones in

the horizontal courses together more firmly. In this case, however, its action in the vertical planes increases the horizontal thrust of the dome towards the exterior, in the direction of the radii, and the flatness may be such as to cause the evil arising from the tendency in the latter direction, to exceed the advantage gained by the former. If, however, the foot of the dome is sufficiently secured against giving away by the horizontal thrust outward, then the dome may be considered more firm in proportion as the curve falls within that of equilibration; within this limit any curve, whether convex or concave, may be chosen for the vertical section of a dome.

Fig. 2, Plate 4, exhibits the curve of equilibration for one half of a dome, and is constructed from a table calculated by Dr. Robison.

It is evident that the voussoirs of domes being bound together by forces acting both in vertical and horizontal planes, form a building, having greater stability than a cylindrical vault, in which the voussoirs are only pressed together in vertical planes: perforation may be made in any part of a dome without sensible injury, the upper part may be either left quite open as in the Pantheon, or it may be crowned by a

cupola, as in the cathedrals of St. Peter and St. Paul.

Dr. Robison shews that when a dome is spherical, it is not safe to employ a segment of more than about 103 degrees for the vertical section: beyond this limit the lower part would fall within the lines of direction of the forces.

TO DETERMINE THE HORIZONTAL THRUST OF A DOME AT ITS BASE.

If the courses of which a dome is composed be indefinitely thin, or, at least, if their thickness bears but a small proportion to the magnitude of the dome, and if the directions of the pressures of the courses in a vertical plane be perpendicular to the joints, as in the dome of equilibration, then it is evident that the pressure upon the lowest joint all round the base of the dome will be the same as would be produced by a cone of equal weight, whose slant side is coincident with the direction of a tangent to the curve of the dome in a vertical plane at the springing course.

Let the curve V A, Fig. 3, Plate 4, be that by

whose revolution about V B the given dome is formed, and let A C be a tangent to V A at the springing. Then, if the cone formed by the revolution of A C be of equal weight with the dome, the pressure of the cone will be equal to that of the dome.

Now, let the whole weight of the dome or cone be accumulated in one point of the side of the cone, suppose at C; then the weight will be to the horizontal thrust at A, produced by that weight, as C B to A B.

If for example, the whole weight of the dome should be 1000 tons, and that A B should be equal to the half of C B, the horizontal thrust at A would be equal to 500 tons; but this thrust is distributed all round the base of the dome, and we are to find what it is equivalent to on every point of that base.

Since A B may be taken to represent the thrust of the dome, we may consider a line equal in length to A B, as expressive of a force which would resist that thrust when applied at one point; and, we may consider the whole thrust as equally diffused over a line equal to A B. Consequently the pressure which each point in such a line would have to support, will

diminish as the length of the line increases; that is, the pressure on any point would be inversely proportional to the length of the line. Therefore, if we suppose the thrust of the dome to be resisted by a force acting all round the base, and tending every where towards the centre, the thrust upon each point of the line equal to A B, will be to that on each point of the base inversely, as that line is to the circumference of the base: that is inversely as the semi-diameter of a circle is to its circumference. But the radius being 1, the circumference is 6·283. Consequently, to find the horizontal thrust on every point of the base in the above example, we may say 6·283 : 1 :: 500 tons : 79½ tons, nearly the thrust required.

If the dome be supported on a circular wall of masonry, whose height is A G, the above thrust must be multiplied by the arm A G of the lever, at the end of which it acts, and the product will express the force with which the dome endeavours to overset the wall. This, of course, must be resisted, as in the case of common vaulting, by a force expressed by the area of the section A D of the wall, multiplied by the half breadth D H (supposing the wall and dome constructed of materials having the same

specific gravity, and the pier rectangular), and as all the terms are given, except D G, this may also be found.

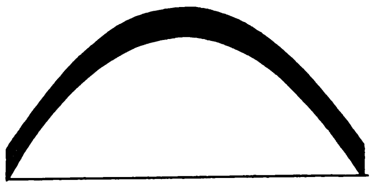
If the dome is intended to be hooped with iron at the base as usual, and we would give such dimensions to the ring that it shall resist the strain, we have only to find from the tables that are published, how much a bar of iron of given dimensions (whose section is one square inch for example), will support, without breaking, when a load is diffused uniformly over it; then the horizontal thrust divided by this tabular number, will give the number of square inches in a section of the intended ring.

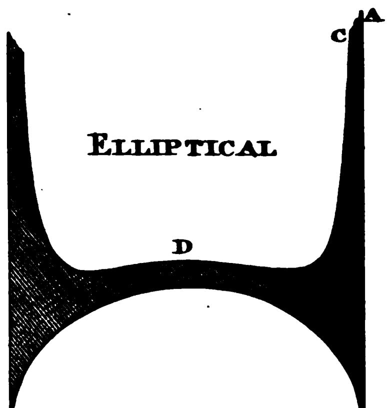
THE END.

G. Woodfall, Printer,
Angel Court, Skinner Street, London.

Plate 1

PARABOLIC





When the intrados is an entire Semi ellipsis AB is infinite being an asymptote to the extrados CD.

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Plate 2.

CATENARÉAN.

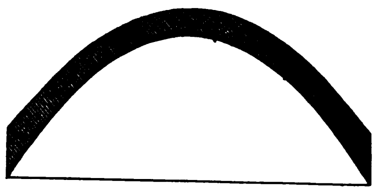

FIG. A.

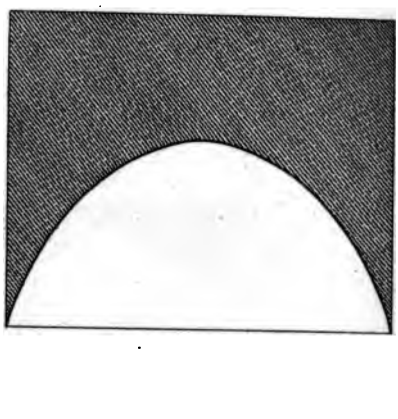

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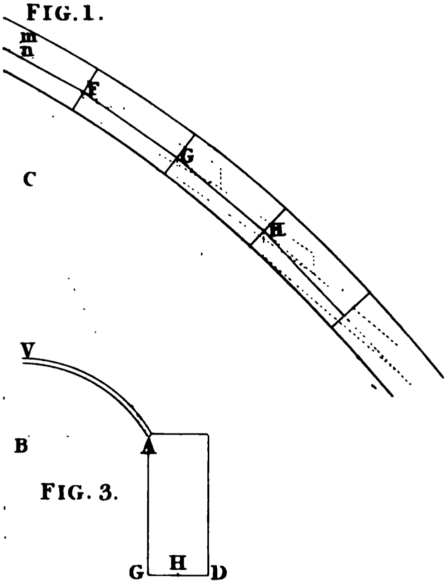
Plate 1.



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M. J.